Heterogeneous Risk Preferences in Financial Markets

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Abstract

This paper builds a continuous time model of a complete financial market with N heterogeneous agents whose CRRA preferences differ in their level of risk aversion. I find that agents dynamically self select into one of three groups depending on their preferences: leveraged investors, diversified investors, and saving divestors, driven by a wedge between the market price of risk and the risk free rate. The model is able to replicate a high market price of risk and a low risk-free rate by separating the markets for risky and risk free assets. This provides an explanation for the equity risk premium and risk-free rate puzzles, while avoiding a preference for early resolution of uncertainty inherent in non-seperable preferences. Additionally, I find that changing the number of preference types has a non-trivial effect on the solution. Finally, I show through numerical solution that the model predicts a correlation between dividend yields and the SDF, a non-linear response of volatility to shocks, and both pro- and counter-cyclical leverage cycles depending on the assumptions about the distribution of preferences.

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I would like to thank my advisors Nicolas Cœoeurdacier and Stéphane Guibaud, as well as Georgy Chabakauri, Suresh Sundaresan, Daniel Lacker, Ronnie Sircar, Jakšă Cvitaniĉ, and Semyon Malamud for helpful comments. I would also like to thank participants at the Columbia GSB Doctoral Seminar, the AEA Annual Meeting Poster Session, the EUROFIDAI December Finance Meeting, the SIAM FME Conference, the IHP Doctoral Game Theory Seminar, the Sciences Po Internal Lunch Seminar, the Berlin Conference on Mathematical Finance for Young Researchers, and the Bachelier Conference on Monte Carlo methods for comments and suggestions. Finally, I would like to thank Nicolò Dalvit, Téo Firpo, Simon Fredon, Léopold Biardeau, Max Viskanic, Edoardo Giscato, and Sarah Kislingbury for lively debate and discussion. A portion of this work was funded by an Alliance Doctoral Mobility Grant.

Introduction

Each day, trillions of dollars worth of financial assets change hands. A financial security gives its bearer the right to a stream of future dividends and capital gains for the infinite future. The price of this stream of dividends is so difficult to determine that if you ask two analysts for an exact price they will generally disagree. This fact has been well documented in studies such as Andrade et al. (2014) or Carlin et al. (2014). However, these observations are in direct contrast to a representative agent model of financial markets. Take for instance the aspect of trade in financial assets previously mentioned. In a representative agent model there can be no exchange because there is no counter party. One looks for a set of prices to make the representative agent indifferent to consuming everything, holding the entire capital stock, etc. In order to have exchange in an economic model we must introduce two or more agents who are heterogeneous in some way. However the type and degree of heterogeneity differs greatly across the literature.

In this paper I focus on heterogeneity in risk preferences and consider how the degree of heterogeneity affects model predictions. In particular I characterize equilibrium in a model with an arbitrary number of preference types and look at comparative statics across models for different assumptions about types. Many authors have studied the problem of heterogeneous preferences under two preference types (e.g. Dumas (1989); Coen-Pirani (2004); Guvenen (2006); Bhamra and Uppal (2014); Chabakauri (2013, 2015); Gârleanu and Panageas (2015); Cozzi (2011)). However, there are few articles which study many types. My work most closely resembles that of Cvitanić et al. (2011), who study an economy populated by N agents who differ in their risk aversion parameter, their rate of time preference, and their beliefs. However, those authors focus on issues of long run survival. I build on their results by studying how changes in the distribution of preferences affect the short run dynamics of the model, while focusing on a single aspect of heterogeneity.[1] Finally, I characterize a Markovian equilibrium in a single state variable, in the style of Chabakauri (2015). This allows me to study how

[1] Additionally, in the appendix I characterize the limit as $N \to \infty$.

financial variables evolve over the entire state space.

Heterogeneous risk preferences are able to match both the market price of risk and the risk free rate, providing a possible explanation for several asset pricing puzzles. Mehra and Prescott (1985) note that under a representative agent with CRRA preferences one needs a very high level of risk aversion to match the equity risk premium, but this implies a high risk-free rate, a phenomenon known as the equity risk premium puzzle. Weil (1989) pointed out that if one attempts to solve this issue by introducing Epstein-Zin preferences, the representative agent must have a very high elasticity of inter-temporal substitution, known as the risk-free rate puzzle. In addition, Epstein et al. (2014) recently added to the controversy by showing that an Epstein-Zin representative agent must be willing to pay a very large amount for the early resolution of uncertainty, implying that solving one puzzle with these preferences creates another. In the model presented here, agents have constant relative risk aversion (a fact noted empirically in Brunnermeier and Nagel (2008)) and are thus indifferent to the resolution of uncertainty. Despite this, the model is still able to produce a high equity risk premium and low risk-free rate by separating the markets for risky and risk-free assets.

When agents exhibit heterogeneous preferences the markets for risky and risk-free assets do not clear at the same preference level. These two assets provide two different services to the individual, one the risk-free transfer of consumption across time and the other the chance to grow their consumption. In fact section 2.3 shows how the two markets overlap, producing three groups of agents who desire to grow only their risky or risk-free asset holding if they are on the extremes of the distribution, or to grow their holdings in both asset types if they lie in the center of the distribution. The interest rate can be low relative to the market price of risk when consumption is skewed towards risk averse agents. These agents are net lenders and desire to grow their bond holdings, implying a relatively large demand for bonds and a low supply, which produces a high price for bonds (or a low interest rate). In fact this relationship can be counter-cyclical as negative shocks will shift consumption towards the risk averse, producing a quantitative implication that the equity risk premium should be negatively correlated with asset prices.

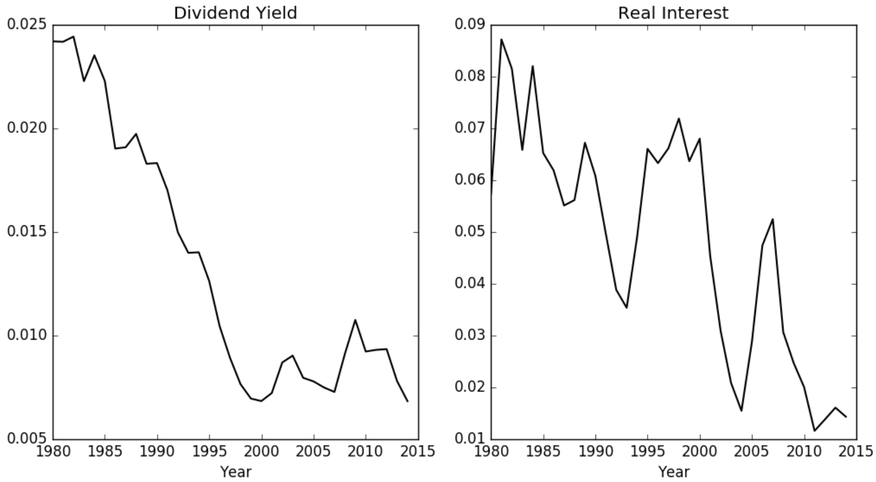

Fig. 1. Dividend yields and real interest rates for U.S. since 1980. Source: Dividend yield from FRED. Real interest rate from IMF, GDP deflator implied inflation.

In addition to these two asset pricing puzzles, the model matches observations about the predictability of returns. In the long run the most risk neutral agent dominates, which produces downward trends in key financial variables. The risk-free rate and dividend yield are negatively correlated with total dividends, producing similar patterns as those seen since the early 1980's in the US (see Figure 1). As GDP has grown, both dividend yield and the risk-free rate have fallen. A negative correlation implies that stock returns have a predictable component. This result is consistent with those of Campbell and Shiller (1988b) and Campbell and Shiller (1988a), who find that the returns on stocks can be predicted as a function of dividend yield. This finding could also explain the result of Mankiw (1981), who rejects the permanent income hypothesis on the basis that asset price co-movements with the stochastic discount factor (SDF) are forecast-able. The SDF does not correspond to a specific agent in every period when agents have heterogeneous risk aversion, but to a time varying preference level. This level is inversely correlated with dividends: a rise in dividends implies a fall in the market price of risk and in the interest rate, the two key variables determining the SDF.

The model also relates to the literature on limited participation and beliefs-driven leverage cycles. The risk free rate and the market price of risk resemble greatly those in Basak and Cuoco (1998). In that paper, asset prices are determined through an exogenous participation constraint. In the present paper the clearing markets for stocks and bonds do not have to correspond to the same agent, nor does the corresponding agent even need to exist in the economy, contrary to the limited participation literature. Additionally, heterogeneity in risk preferences can produce both pro- and counter-cyclical leverage cycles, contrary to the beliefs-driven leverage cycle literature (e.g. Geanakoplos (2010)[2]). This fact is driven by two factors: complete markets and the volatility of the supply of credit. Given that markets are complete, a negative shock causes all agents to increase their portfolio weights, but the same shock generates a fall in wealth. The combined effect produces a heterogeneous response in the value of agents' borrowing. Risk neutral agents reduce the value of their borrowing while risk averse agents increase borrowing (or dis-save), albeit to a lesser degree. The net result is a reduction in the aggregate supply of credit. However, asset prices fall simultaneously and to a greater degree. Total leverage rises, given that the value of borrowing falls less than the value of equity. In order to produce a pro-cyclical leverage cycle as seen in Geanakoplos (2010), one must either introduce constraints or consider a distribution of preferences such that a very small group borrows from a deep pool of lenders.

Finally the model relates to the study of models with a large number of heterogeneous agents, otherwise known as Mean Field Games. Games featuring a continuum of agents hearken back to Aumann (1964). However, their study in a stochastic setting has recently garnered a large amount of attention thanks to a series of papers by Jean-Michel Lasry and Pierre-Louis Lions (Lasry and Lions (2006a), Lasry and Lions (2006b), Lasry and Lions (2007)). These authors studied the limit of N-player stochastic differential games as $N \to \infty$ and agents' risk is idiosyncratic, dubbing the system of equations governing the limit a "Mean Field Game" (MFG). Their work has then been applied to macroeconomics in works such as Moll (2014), Achdou

[2]I also should thank John Geanakoplos, since his work inspired this paper.

et al. (2014), and Kaplan et al. (2016). However, these papers focus on idiosyncratic risk and do not study the problem of aggregate shocks, which is a lively area of research (e.g. Carmona et al. (2014), Carmona and Delarue (2013), Chassagneux et al. (2014), and Cardaliaguet et al. (2015)) and which encompass classic macroeconomic models such as Krusell and Smith (1998). The approach is either to use a stochastic Pontryagin maximum principle to derive a system of forward-backward stochastic differential equations governing the solution, or to define an infinite dimensional PDE governing the agents' value functions. These approaches are clear from a mathematical perspective, but very difficult to formulate for more complex economic models (although Ahn et al. (2016) provides a method for approximating the solution using the latter method).

This paper takes a different approach, solving the model with common noise using Girsanov theory in the style of Harrison and Pliska (1981) and Karatzas et al. (1987). One then recognizes that the SDF can be written as a function of a single state variable. This implies a Markovian equilibrium in the style of Chabakauri (2013) and Chabakauri (2015). The solution is characterized by mean field dependence through the control, as opposed to the state. This points towards a new way to consider control in mean field financial models. If the stochastic discount factor can be written as a function of a small number of state variables, then the atomistic agents do not need to consider the entire distribution of individual states in order to solve their problem. This result is driven by complete markets, as ratios of marginal utilities are constant, and requires further study for incomplete markets.

The paper is organized as follows: in Section 1, I construct a continuous time model of financial markets populated by a finite number of agents who differ in their preferences towards risk. Section 2 characterizes the equilibrium. Section 3 provides numerical results. Section 4 concludes. The more technical analysis and proofs have been relegated to the appendix.

1. The Model

Consider a continuous time Lucas (1978) economy populated by a number, N, of heterogeneous agents indexed by $i \in \{1, 2, ..., N\}$. Each agent has constant relative risk aversion (CRRA) preferences with relative risk aversion γ_i:

$$U_i(c_{it}) = \frac{c_{it}^{1-\gamma_i}}{1-\gamma_i} \qquad \forall i \in \{1, 2, ..., N\}$$

Agents begin with a possibly heterogeneous initial share in risky assets, denoted α_i[3]. Agents' initial conditions are distributed according to a density $(\gamma_i, \alpha_i) \sim g(\gamma, \alpha)$.

Agents can continuously trade in shares of a per-capita dividend process, which follows a geometric Brownian motion (GBM):

$$\frac{dD_t}{D_t} = \mu_D dt + \sigma_D dW_t \tag{1}$$

where μ_D and σ_D are constants. Risky share prices and risk-free bond prices follow an Itô process and an exponential process, respectively:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t \tag{2}$$

$$\frac{dS_t^0}{S_t^0} = r_t dt \tag{3}$$

Denote by X_{it} an individuals' wealth at time t and by π_{it} the share of an individual's wealth invested in the risky stock. These assumptions imply that an agent's initial wealth is defined by $X_{i0} = \alpha_i S_0$.

[3]Assume for simplicity that initial holdings on bonds is zero.

1.1. Budget Constraints and Individual Optimization

An individual agent's constrained maximization subject to instantaneous changes in wealth can be written as:

$$\begin{aligned}
\max_{\{c_{it},\pi_{it}\}_{t=0}^{\infty}} \quad & \mathbb{E}\int_0^{\infty} e^{-\rho t}\frac{c_{it}^{1-\gamma_i}}{1-\gamma_i}dt \\
\text{s.t.} \quad & dX_{it} = \left[X_{it}\left(r_t + \pi_{it}\left(\mu_t + \frac{D_t}{S_t} - r_t\right)\right) - c_{it}\right]dt \\
& + \pi_{it}X_{it}\sigma_t dW_t
\end{aligned}$$

where the constraint represents the dynamic budget of an individual.

1.2. Equilibrium

Each agent will be considered to be a price taker. This implies an Arrow-Debreu type equilibrium concept.

Definition 1. *An equilibrium in this economy is defined by a set of processes* $\{r_t, S_t, \{c_{it}, X_{it}, \pi_{it}\}_{i=1}^{N}\}$ $\forall$ t*, given preferences and initial endowments, such that* $\{c_{it}, X_{it}, \pi_{it}\}$ *solve the agents' individual optimization problems and the following set of market clearing conditions is satisfied:*

$$\frac{1}{N}\sum_i c_{it} = D_t \ , \ \frac{1}{N}\sum_i (1-\pi_{it})X_{it} = 0 \ , \ \frac{1}{N}\sum_i \pi_{it}X_{it} = S_t \tag{4}$$

I consider Markovian equilibria where the problem can be written as a function of some finite number of state variables.

2. Equilibrium Characterization

To solve this problem I first use the martingale method to show how the SDF can be written as a function of a single state variable. I then use the Hamilton-Jacobi-Bellman (HJB) equation to derive a system of ordinary differential equations which determine the portfolio and stock price volatility.

2.1. The Static Problem

Following Karatzas and Shreve (1998) define the SDF, H_t, as

$$\frac{dH_t}{H_t} = -r_t dt - \theta_t dW_t \qquad \textit{where} \qquad \theta_t = \frac{\mu_t + \frac{D_t}{S_t} - r_t}{\sigma_t} \tag{5}$$

Here θ_t represents the market price of risk. Following Proposition 2.6 from Karatzas et al. (1987), we can rewrite each agent's dynamic problem as a static one beginning at time $t = 0$

$$\begin{aligned} \max_{\{c_{it}\}_{t=0}^{\infty}} \quad & \mathbb{E} \int_0^\infty e^{-\rho t} \frac{c_{it}^{1-\gamma_i}}{1-\gamma_i} dt \\ \text{s.t.} \quad & \mathbb{E} \int_0^\infty H_t c_{it} dt \leq x_i \end{aligned}$$

If we denote by Λ_i the Lagrange multiplier in individual i's problem, then the first order conditions can be rewritten as

$$c_{it} = \left(e^{\rho t} \Lambda_i H_t\right)^{\frac{-1}{\gamma_i}} \tag{6}$$

which holds for every agent in every period.

Given each agent's first order conditions, we can derive an expression for consumption as a fraction of per-capita dividends.

Proposition 1. *One can define the consumption of individual, i, at any time, t, as a share ω_{it} of the per-capita dividend, D_t, such that*

$$c_{it} = \omega_{it} D_t \quad \textit{where} \;\; \omega_{it} = \frac{N\left(\Lambda_i e^{\rho t} H_t\right)^{\frac{-1}{\gamma_i}}}{\sum_j \left(\Lambda_j e^{\rho t} H_t\right)^{\frac{-1}{\gamma_j}}} \tag{7}$$

This expression recalls the results in Basak and Cuoco (1998) or Cuoco and He (1994), where ω_{it} acts like a time-varying Pareto-Negishi weight. In those works, however, this weight arises from an imperfection in the information structure or some exogenous constraint. Here the weights come from a choice of participation driven by preferences towards risk and are completely endogenous. The weight an agent gives to the SDF differs depending on the agents' risk aversion, despite the value of the SDF being equal across agents. This observation leads one to think that perhaps it would be better to think of this as an incomplete market. If markets were fully complete, there would be a risky asset for each agent, but here agents are forced to bargain over a single asset.

The following lemma is useful to derive an expression for the risk free rate and the market price of risk and sheds light on individual consumption patterns:

Lemma 1. *An agent's consumption follows an Itô process with drift and diffusion coefficients* $c_{it}\mu_{it}^c$ *and* $c_{it}\sigma_{it}^c$ *such that*

$$\begin{aligned} r_t &= \rho + \mu_{it}^c \gamma_i - (1+\gamma_i)\gamma_i \frac{(\sigma_{it}^c)^2}{2} \\ \theta_t &= \sigma_{it}^c \gamma_i \end{aligned}$$

which hold simultaneously for all i*.*

These formulas resemble those one would find in a standard representative agent model. However, these expressions hold simultaneously for all agents. Shocks cause the growth rate and volatility of consumption for each agent to adjust, while for a representative agent they would be replaced by the drift and diffusion of the dividend process. Rewrite Lemma 1 in terms of μ_{it}^c and σ_{it}^c and differentiate in order to better understand how these values adjust:

$$\frac{\partial \mu_{it}^c}{\partial \theta_t} = \frac{1+\gamma_i}{\gamma_i^2}\theta_t \tag{8}$$

$$\frac{\partial \mu_{it}^c}{\partial r_t} = \frac{1}{\gamma_i} \tag{9}$$

$$\frac{\partial \sigma_{it}^c}{\partial \theta_t} = \frac{1}{\gamma_i} \tag{10}$$

$$\frac{\partial \sigma_{it}^c}{\partial r_t} = 0 \tag{11}$$

The derivatives of individual consumption parameters imply heterogeneity in the size of response to changes in financial variables, but homogeneity in sign. Eqs. (8) and (9) imply that the growth rate of every individual's consumption is increasing in both the market price of risk and in the interest rate. A higher market price of risk implies greater returns. These returns mean any given agent earns more on their portfolio and a wealth effect dominates. In regards to interest rates, agents choose their consumption growth conditional on their consumption at time t. Thus any change in the interest rate must pivot their life-time budget constraint through this point. This produces a positive and dominant wealth effect whether an agent is a net-borrower or -lender, as an increase in the interest rate makes life-time consumption less costly. In addition, an increase in the market price of risk increases volatility of consumption for all agents, while the interest rate has no effect (Eqs. (10) and (11)). A higher market price of risk implies a greater volatility (in absolute value) of the SDF. Because of this, the present value of discounted future consumption becomes more volatile and in turn consumption becomes more volatile.

2.2. The Risk-Free Rate and Market Price of Risk

Given Lemma 1, we can derive expressions for the market price of risk and the risk free rate:

Proposition 2. *The interest rate and market price of risk are fully determined by the sufficient statistics* $\xi_t = \frac{1}{N}\sum_i \frac{\omega_{it}}{\gamma_i}$ *and* $\phi_t = \frac{1}{N}\sum_i \frac{\omega_{it}}{\gamma_i^2}$ *such that*

$$r_t = \rho + \frac{\mu_D}{\xi_t} - \frac{1}{2}\frac{\xi_t + \phi_t}{\xi_t^3}\sigma_D^2 \tag{12}$$

$$\theta_t = \frac{\sigma_D}{\xi_t} \tag{13}$$

Proposition 2 is in terms of only certain moments of the joint distribution of consumption shares and risk aversion: $\xi_t = \frac{1}{N}\sum_i \frac{\omega_{it}}{\gamma_i}$ and $\phi_t = \frac{1}{N}\sum_i \frac{\omega_{it}}{\gamma_i^2}$. These moments represent weighted averages of elasticity of intertemporal substitution (EIS). An agent's preferences only affect the market clearing interest rate and market price of risk up to their share in consumption.

The expressions in Proposition 2 are similar to those one would find in a representative agent economy. The market price of risk is equal to that which would prevail in a representative agent economy populated by an agent whose elasticity of inter-temporal substitution is equal ξ_t. The interest rate is reminiscent of the interest rate in an economy populated by the same agent, but not quite equal. We could rewrite Eq. (12) as the interest rate that would prevail in our hypothetical economy plus an extra term:

$$r_t = \rho + \frac{\mu_D}{\xi_t} - \frac{1}{2}\frac{\xi_t + 1}{\xi_t^2}\sigma_D^2 - \mathbf{\frac{1}{2}\frac{1}{\xi_t}\left(\frac{\phi_t}{\xi_t^2} - 1\right)\sigma_D^2}$$

This additional term (in bold) would be zero if $\phi_t = \xi_t^2$. However, we can apply the discrete version of Jensen's inequality to show that $\phi_t > \xi_t^2, \quad \forall t < \infty$. This causes the additional term to be strictly negative. The risk free rate is then lower than it would be in an economy populated by a representative agent. This introduces a sort of "heterogeneity wedge", which I'll define as $\frac{\phi_t}{\xi_t^2} > 1$, between the price of risk and the price for risk free borrowing. This wedge is also equal to one plus the squared coefficient of variation of the weighted EIS. The wedge will be higher when the variation in the EIS is higher, weighted by consumption shares. The driving force behind the heterogeneity wedge is the separation between the marginal agents in the markets for risky and risk-free assets that occurs when agents differ in their preferences towards risk.

2.3. Equity Risk Premium and Marginal Agents

This subsection discusses a comparison to partial equilibrium to better explain the intuition for why the market price of risk can be high and interest rate low when preferences are heterogeneous. The markets for risky and risk free assets do not clear at the same level and generate three distinct groups. Define $\{\gamma_{rt}, \gamma_{\theta t}\}$ to be the RRA parameters in a representative agent economy that would produce the same interest rate and market price of risk, respectively. These preference levels can be interpreted as the marginal agent in each market and are given by

$$\gamma_{rt} = \frac{\mu_D}{\sigma_D^2} - \frac{1}{2} - \sqrt{\frac{\mu_D}{\sigma_D^2}\left(\frac{\mu_D}{\sigma_D^2} - 1 - \frac{2}{\xi_t}\right) + \frac{\xi_t + \phi_t}{\xi_t^3} + \frac{1}{4}}$$

$$\gamma_{\theta t} = \frac{1}{\xi_t}$$

It can be shown that $\gamma_{rt} < \gamma_{\theta t}$, $\forall t < \infty$. This observation implies that the markets for risky and risk-free assets do not coincide in finite t and that the two markets overlap (see Figure 2). This overlap creates three groups of agents: leveraged investors, diversifying investors, and saving divestors.

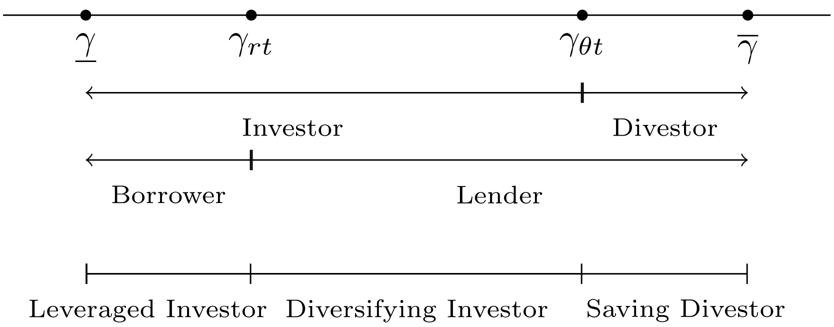

Fig. 2. An agent's position relative to γ_{rt} and $\gamma_{\theta t}$ shed light on their consumption and saving decisions.

The three groups represent buyers and sellers of risky and risk-free assets. Low risk aversion agents sell bonds in order to buy a larger share in the risky asset. High risk aversion agents purchase bonds and shrink their share in the risky asset. These agents are exchanging with the low risk aversion agents their risky shares for bonds. In the middle agents purchase both bonds and shares in the risky asset. They do this by capitalizing their gains in the risky asset. These relationships cause the risky asset to be concentrated among the low risk aversion investors as the economy grows and pushes up asset prices. At the same time, the overlap between these two markets causes the marginal agent in the market for risk free bonds to be more risk neutral, reducing interest rates. Low interest rates relative to the market price of risk creates an increase in the equity risk premium.

Variance in the distribution of preferences can explain the equity risk premium puzzle and risk-free rate puzzle of Mehra and Prescott (1985) and Weil (1989). Consider fixing the time $t = 0$ interest rate[4] to be $r_0 = 0.03$ and looking to match different values of θ_0. This implies different values of ξ_0 and ϕ_0, which are represented in Figure 3(a) for different parameter values of μ_D. This figure shows that a key parameter in matching both the interest rate and market price of risk is the weighted variance in EIS, ϕ_t. Consumption shares must be concentrated among the risk neutral to produce a low ξ_t and thus a high θ_0, but there must also be high variance in preferences in order to have a low risk-free rate. This can be explained through a supply and demand argument, as the variance in the distribution of preferences determines both the supply and demand for bonds. Increasing the variance while keeping the mean of the distribution fixed requires skewing consumption weights towards the high risk aversion agents, as their EIS is very low. Risk averse agents are net lenders, as indicated by Figure 2, so this higher variance shifts out the demand for bonds and shifts in the supply. This causes an increase in the price of bonds and, in turn, a fall in the interest rate. Heterogeneity in preferences can in this way produce a low interest rate and a high market price of risk depending on where consumption weights are concentrated.

Figure 3(b) shows the consumption weights necessary to match the same r_0 and θ_0 as Figure 3(a) when $\mu_D = 0.01$. We can note first that this set of preferences cannot match both a low risk free rate and low market price of risk[5] except for a very narrow set of values. Next, the share of the agent in the middle of the distribution of preferences is increasing as we increase the market price of risk. This phenomenon is related to the point made earlier, that skewing consumption weights towards the more risk averse produces a low risk-free rate and high market price of risk. Preference heterogeneity beyond two preference types allows one the additional free variable necessary to match both the risk free rate and market price of risk, without introducing

[4]The values of r_0 and preference parameters (to follow) are chosen such that the solution exists and the extreme interest rate is non-negative. Other parameters are fixed to $(\sigma_D, \rho) = (0.032, 0.02)$.

[5]...indicated by a negative value for one of the consumption weights...

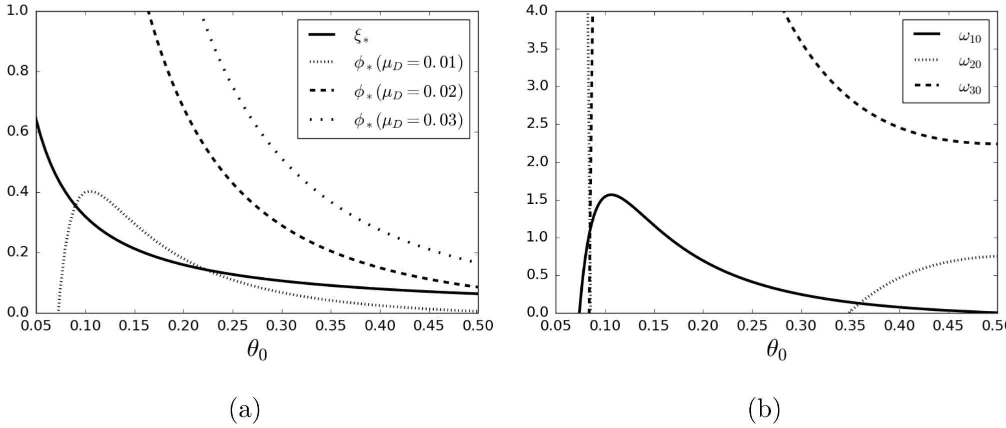

Fig. 3. Results regarding the equity risk premium depend on the parameters. Holding $r_0 = 0.03$ fixed, Figure 3(a) depicts the values of ξ_0 and ϕ_0 which match different values of θ_0 and shows the difference assuming different parameters for μ_D. Figure 3(b) performs the same exercise for ω_{i0} holding $\mu_D = 0.01$ fixed and assuming 3 agents with preference parameters $(\gamma_1, \gamma_2, \gamma_3) = (1.1, 10, 18)$.

a preference for early resolution of uncertainty.

In Epstein et al. (2014), it is shown that in order to match the equity risk premium using Epstein-Zin preferences, one must introduce a strong preference for the early resolution of uncertainty. In fact, the authors show that the representative agent would have to be willing to pay an exorbitant amount to resolve the uncertainty. In the model presented here, agents are CRRA and, thus, indifferent to early or late resolution of uncertainty. High expected returns in this model are driven in part by agents preferences, and in part by the dynamics of their consumption weights which can be derived explicitly.

2.4. Consumption Weight Dynamics

We can study the dynamics of an agent's consumption weight by applying Itô's lemma to the expression given in Proposition 1.

Proposition 3. *Assuming consumption weights follow an Itô process such* *that*

$$\frac{d\omega_{it}}{\omega_{it}} = \mu_{it}^{\omega} dt + \sigma_{it}^{\omega} dW_t$$

an application of Itô's lemma to (7) *gives expressions for* μ_{it}^{ω} *and* σ_{it}^{ω}*:*

$$\mu_{it}^{\omega} = (r_t - \rho)\left(\frac{1}{\gamma_i} - \xi_t\right) \tag{14}$$

$$+ \frac{\theta_t^2}{2}\left[\left(\frac{1}{\gamma_i^2} - \phi_t\right) - 2\xi_t\left(\frac{1}{\gamma_i} - \xi_t\right) + \left(\frac{1}{\gamma_i} - \xi_t\right)\right]$$

$$\sigma_{it}^{\omega} = \theta_t\left(\frac{1}{\gamma_i} - \xi_t\right) \tag{15}$$

Individual consumption weights evolve as functions of ξ_t and ϕ_t. Consider first the case where an agent's preferences coincide with the weighted average, ie $\gamma_i = \gamma_{\theta t} = \frac{1}{\xi_t}$ (as in section 2.3). In this case ω_{it} is instantaneously deterministic, i.e. $\sigma_{it}^{\omega} = 0$. This determinism arises because the agent represents the marginal agent in the market for risky assets. However, notice that in this case $\mu_{it}^{\omega} = \theta_t^2\left(\frac{1}{\gamma_{\theta t}^2} - \phi_t\right) = \sigma_D^2\left(1 - \frac{\phi_t}{\xi_t^2}\right)$. The agent is moving deterministically out of the marginal position. The speed with which this is occurring is driven by the heterogeneity wedge, $\frac{\phi_t}{\xi_t^2}$. When this wedge is high, the rate at which the marginal agent moves out of the marginal position is greater.

Next consider the case where an agent is more patient than the weighted average, that is $\gamma^i > \gamma_{\theta t}$. Then $\sigma_{it}^{\omega} < 0$ and agent i's weight is negatively correlated to the market. This negative correlation implies that if an agent is more patient than the average, or alternatively more risk averse, then their consumption share increases when there are negative shocks and decreases when there are positive shocks. This co-movement can be thought of as playing a "buy low, sell high" strategy for consumption. Conversely, risk neutral agents shares co-vary positively with the market, i.e. $\sigma_{it}^{\omega} > 0$. These risk neutral agents are impatient and value present dividends over future dividends. A shock to the dividend process shifts the level permanently because of the martingale property of the Brownian motion. A fall in the level makes agents poorer today and in the near future, so for these impatient agents the income effect dominates and their lifetime income shifts substantially. To compensate they must shift consumption. These are the day-traders, riding booms and busts to try to make a quick buck while not losing their shirts. Although they may benefit in the short run, their consumption is more volatile than the economy.

2.5. Asset Prices and Portfolios

Asset prices and portfolios can be derived via a combination of the HJB and the martingale method. In fact it can be shown (and verified) that the individual maximization problem can be formulated as a function of only two state variables, individual wealth and the dividend process. Recall the first order condition Eq. (6) and substitute this into the market clearing condition for consumption:

$$\frac{1}{N}\sum_i \left(e^{\rho t} H_t\right)^{\frac{-1}{\gamma_i}} = D_t$$

This shows that the sdf is an implicit function only of the dividend process D_t and time. Given this, it is natural to look for a solution to an agent's maximization in the two state variables (X_{it}, D_t) and time. However, given the homogeneity of CRRA preferences, the value functions factor and the dependence on wealth and time disappears, giving a solution in a single state variable. This solution is given in Proposition 4:

Proposition 4. *Assuming there exists a Markovian equilibrium in D_t, the individuals' wealth-consumption ratios, given by $V_i(D_t)$ satisfy ODE's given by*

$$\begin{aligned} 0 =& 1 + \frac{\sigma_D^2 D_t^2}{2} V_i''(D_t) + \left[\frac{1-\gamma_i}{\gamma_i}\theta_t \sigma_D + \mu_D\right] D_t V_i'(D_t) \\ &+ \left[(1-\gamma_i) r_t - \rho + \frac{1-\gamma_i}{2\gamma_i}\theta_t^2\right] \frac{V_i(D_t)}{\gamma_i} \end{aligned} \tag{16}$$

which satisfy boundary conditions

$$V_i(D^*) = \frac{\gamma_i}{\rho - (1-\gamma_i)\left(\frac{\theta(D^*)^2}{2\gamma_i} + r(D^*)\right)} \quad \textit{for } D^* \in \{0, \infty\} \tag{17}$$

while portfolios and volatility are given by:

$$\pi_{it} = \frac{1}{\gamma_i \sigma_t}\left(\gamma_i \sigma_D D_t \frac{V_i'(D_t)}{V_i(D_t)} + \theta_t\right) \tag{18}$$

$$\sigma_t = \sigma_D \left(1 + D_t \frac{\mathcal{S}_t'(D_t)}{\mathcal{S}_t(D_t)}\right) \tag{19}$$

where the price/dividend ratio $\mathcal{S}_t(D_t)$ *satisfies*

$$\mathcal{S}_t(D_t) = \frac{1}{N}\sum_i V_i(D_t)\omega_{it} \tag{20}$$

and where $\omega_{it} = f_i(D_t)$, *such that* $f_i(\cdot)$ *is an implicit function satisfying*

$$\frac{1}{N}\sum_j \lambda_{ji}^{\frac{-1}{\gamma_j}} f_i(z)^{\frac{\gamma_i}{\gamma_j}} z^{\frac{\gamma_j - \gamma_i}{\gamma_j}} = 1 \ \textit{ where } \lambda_{ji} = \frac{\Lambda_j}{\Lambda_i} \tag{21}$$

Several facts jump out from this solution. First, one can notice in Eq. (18) that portfolios separate in two terms as is common in the literature (see Merton (1969, 1971)): a myopic term and a hedging term. The myopic term is the market price of risk scaled by risk preferences and volatilities, representing the portfolio desired at the current point in time. The hedging term is determined by the shape of the agent's wealth consumption ratio and the fact that returns are inversely correlated with the market. A fall in the dividend process causes the more risk averse agents to dominate, reducing asset prices relative to dividends and increasing the dividend yield. Thus, the agent hedges possible losses in dividends with capital gains through their risky shares.

In addition, the model exhibits excess volatility which is inversely correlated to the market, determined by the shape of the price/dividend ratio. In Eq. (19) stock price volatility is the fundamental volatility, σ_D, plus the instantaneous volatility of the dividend scaled by the rate of change in the price/dividend ratio. A fall in the dividend again causes the risk averse agents to dominate, which in turn implies a fall in the price/dividend ratio. Thus the price/dividend ratio has a positive slope everywhere, but it is in fact concave. A negative shock causes this slope to rise, increasing the volatility. This fact is reminiscent of the implied volatility smile inherent in options pricing data (Fouque et al. (2011)).

Given the above proposed equilibrium, following identical steps as Chabakauri (2015), one can show existence of an equilibrium satisfied by the above proposition.

Proposition 5. *Suppose there exist bounded positive functions $V(D) \in C^1[0, \infty) \cup C^2[0, \infty)$ that satisfy the system of ODE's in Eq.* (16) *and boundary conditions in Eq.* (17). *Additionally, assume the processes θ_t, σ_t, and π_{it} are bounded and that $|\sigma_t| > 0$. Then there exists a Markovian equilibrium satisfied by Propostions 2 and 4*

These equations recall the results in Chabakauri (2013, 2015), but characterize the equilibrium in a slightly different way. In particular the dimension remains low by studying an equilibrium taking only the dividend as a state instead of taking consumption weights as the state variable (which would make the dimension of the problem very large). In fact, it is easy to study the limiting case of a continuum of types given that the dimension of the state space does not change with the number of preference types although this has been left to the appendix.

3. Simulation Results and Analysis

In this section, I review the simulation strategy as well as some simulation results. The underlying assumption is that I am attempting to approximate a continuous distribution of types (for a recent survey on estimating risk preferences see Barseghyan et al. (2015) and for evidence on heterogeneity see Chiappori et al. (2012)). One could argue that this approximation is the goal of any model of heterogeneous risk preferences with finite types, as in Dumas (1989), Gârleanu and Panageas (2015), Chabakauri (2015), or

Chabakauri (2013), but that one assumes finite types for tractability and with the hope that the results generalize. This section tries to convince you that results for two types do not necessarily generalize to more types. [6]

For all of the simulations, I hold the following group of parameters fixed at the given values: $\mu_D = 0.01$, $\sigma_D = 0.032$, and $\rho = 0.02$. These settings correspond to a yearly parameterization. Solutions are represented over the state space, $D_t \in [0, \infty)$, which is truncated for clarity as most of the action is in the lower regions. Additionally, assume that all agents begin with an initial wealth such that their consumption weights are equalized at $t = 0$. I present here results for different numbers of types and leave to the appendix continuous types and robustness results.

3.1. Increasing Types

Assume preferences are distributed uniformly over $[1.1, 18.0]$. Consider changing the number of evenly spaced types over the support. I consider the cases where $N = 2, 5, 10$ and 100. Plots are presented for particular variables showing the results for the different settings. The motivation here is to consider whether results for two agents generalize to many types. The change in the number of types is shown to have a significant effect on the level of all aggregate variables. This effect implies that mis-specification of the support of the distribution of risk preferences has a non-trivial effect on the model's short run predictions about market variables.

Consider first the two key pricing variables ξ_t and ϕ_t, as shown in Figure 4. You can immediately notice that the level, slope, and curvature of both of these measures changes substantially for different numbers of preference types. Additionally you can notice that the heterogeneity wedge, defined as $\frac{\phi(t)}{\xi(t)^2}$ is also changing. Changing the mass of agents over the support changes the rate of convergence, as in order for the most risk neutral agent to dominate they must accumulate a greater consumption share to bring ξ_t and ϕ_t to the same long run level.

In Figure 5 you can see that the level, slope, and curvature of the interest

[6] For details on the solution method, see appendix B.

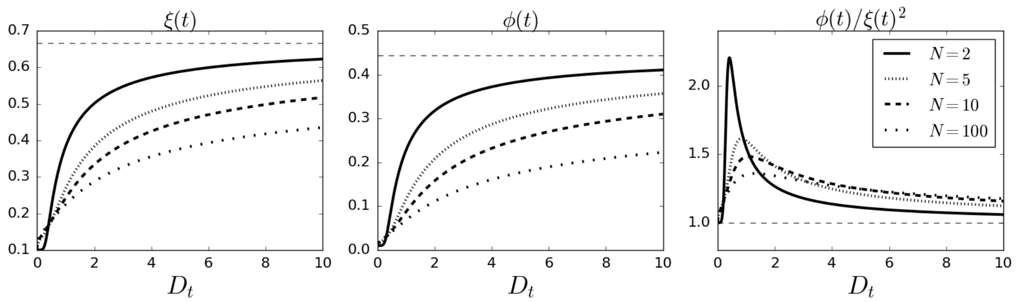

Fig. 4. Sufficient statistics for the distribution of the risk aversion with finite types, where $\xi_t = \frac{1}{N}\sum_{i=1}^{N}\frac{\omega_{it}}{\gamma_i}$ and $\phi_t = \frac{1}{N}\sum_{i=1}^{N}\frac{\omega_{it}}{\gamma_i^2}$. N corresponds to the number of types.

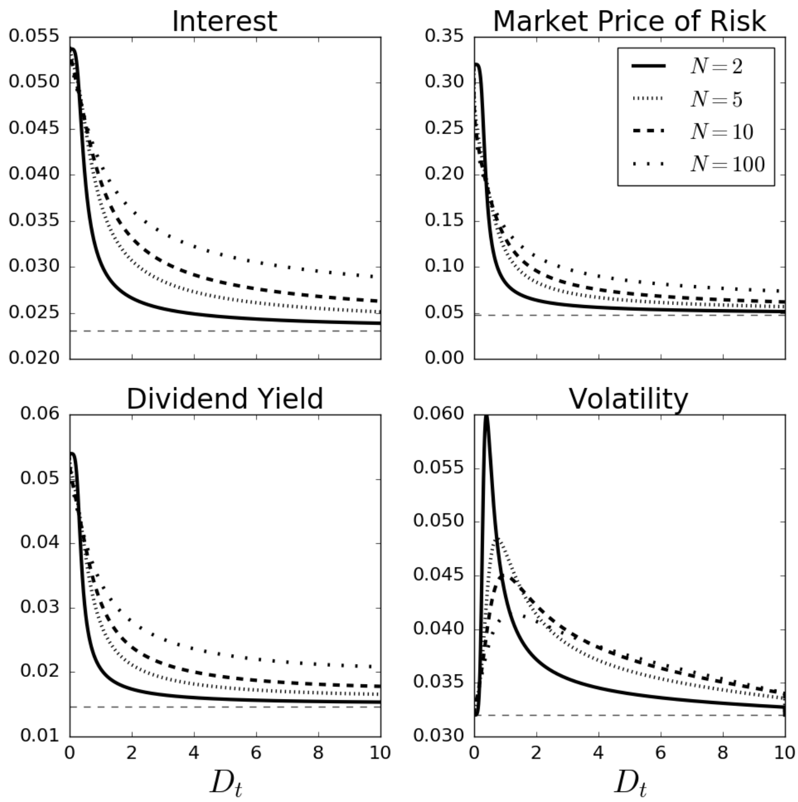

Fig. 5. Interest rate, market price of risk, dividend yield, and volatility for different numbers of agents.

rate and market price of risk are different for different numbers of preference types. For two types the convergence is fast. Both variables rise and their convergence becomes slower as we increase the number of agents, implying a more long-lived volatility in asset pricing variables. This volatility is consistent with what we observe in financial markets, in that the interest rate and market price of risk do indeed vary with output. In addition, the change in levels is substantial, with the interest rate being 100 basis points higher at $D = 2.0$ for 100 agents than for 2 agents.

Changes in the interest rate and market price of risk have noticeable effects on risky asset prices. Asset prices fall in the face of higher levels of the interest rate and market price of risk, while volatility rises (Figure 5). More preference types drives down the average EIS. Meanwhile, more types reduces the speed of convergence, causing small shocks to produce larger changes in asset prices and increasing the volatility. In addition, volatility has a large spike for low values of the dividend when there is less heterogeneity, implying that downside volatility is greater when preference heterogeneity is low. Beyond simply the level, it is interesting to note the slope and curvature of volatility. A negative shock increases volatility, while a positive shock reduces volatility to a lesser degree. This points to one possible explanation to the short-coming often noted in the celebrated work of Black and Scholes (1973), namely constant volatility. The implied volatility estimated from options pricing data exhibits a similar negative slope and convex shape, known as the "volatility smile"[7]. This model produces such stochastic volatility and, in particular, greater downside volatility. In addition to stochastic volatility, the dividend yield co-moves with aggregate consumption, implying predictability in stock market returns as observed in the data by Campbell and Shiller (1988a,b). This is driven by co-movement between the SDF and aggregate consumption, similar to that observed in Mankiw (1981).

On the bond side, we see a non-zero and stochastic supply of credit, creating "leverage cycles". Increasing the number of agents increases leverage, as shown in Figure 6. This is driven by two forces. More preference types means a larger supply of bonds. At the same time, more types reduces asset prices, reducing the value of agents' collateral and further increasing leverage. How-

[7]See Lorig and Sircar (2016) for a nice review or Guyon and Henry-Labordère (2013) for a thorough mathematical treatment of several approaches.

ever, notice that a rise in D_t generates a fall in leverage and vice-versa. This implies a counter-cyclical leverage cycle, opposite that postulated in the literature (e.g. Fostel and Geanakoplos (2008)). This contradiction arises from a difference in the response of borrowing and asset prices. Both are positively correlated with the dividend, but since asset prices are more volatile a fall in the dividend causes a greater fall in asset prices, which increases total leverage.

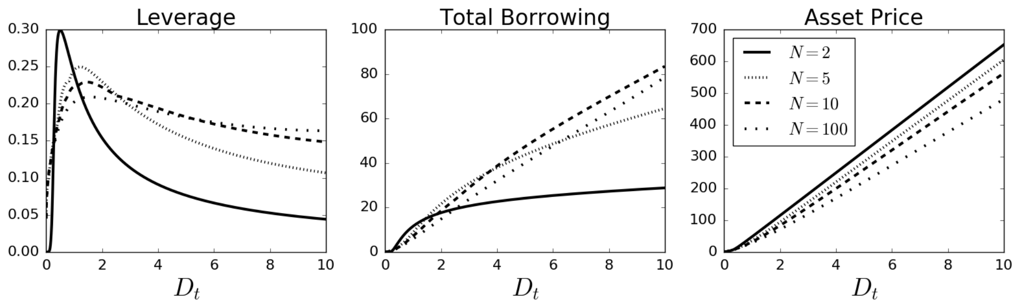

Fig. 6. Leverage, total borrowing, and asset prices for different numbers of agents.

In summation, heterogeneous preferences can explain several qualitative features about financial markets, but the choice of the number of preference types is important if one believes there is a continuum of types.

4. Conclusion

In this paper I have studied how the distribution of risk preferences affects financial variables, consumption shares, and portfolio decisions. The distribution of risk preferences has a large effect on financial variables driven mainly by consumption weighted averages of the EIS. Agents' relative position to these averages determines three groups of market participants. Leveraged investors have low risk aversion and borrow in order to grow their share in consumption. Saving divestors are highly risk averse and lend in order to shrink their share in consumption. Diversifying investors lie in the middle, simultaneously growing their share in consumption and lending by buying bonds.

The model can produce both pro- and counter-cyclical leverage cycles depending on the distribution of preferences. The cyclicality of the leverage cycle is driven by the volatility of total borrowing and asset prices. When total borrowing is more volatile than asset prices, leverage cycles are pro-cyclical, and vice-versa. In order for total borrowing to be volatile, there needs to be a large mass of lenders and a small, wealth-poor, risk neutral group of borrowers.

Additionally, dividend yield in this model co-moves negatively with the growth rate in dividends. This co-movement implies a predictable component in stock market returns. Dividends fall when a negative shock hits the economy and the distribution of consumption shares shifts towards more risk averse agents. This shift reduces asset prices and predicts a faster growth rate in the future. Papers such as Campbell and Shiller (1988a), Campbell and Shiller (1988b), Mankiw (1981), and Hall (1979) drew differing conclusions about the standard model of asset prices, but, broadly speaking, they all deduced that there was some portion of asset prices that was slightly predictable as a function of the growth rate in aggregate consumption. In the model presented here, we can take a step towards explaining this predictability as the dividend yield co-moves with the SDF.

Finally, heterogeneous risk preferences can produce both a low risk-free rate and high market price of risk. This comes from the fact that the marginal agent in the markets for risky and risk-free assets are not the same. By changing the variance in the distribution of preferences, it is possible to skew the distribution in such a way as to match the equity risk premium and the risk free rate. This provides a possible explanation of the equity risk premium puzzle and risk-free rate puzzle of Mehra and Prescott (1985) and Weil (1989), while avoiding the issue of preference for early resolution of risk noted by Epstein et al. (2014).

An interesting direction for future research would be to carry this approach over to incomplete markets, as in Chabakauri (2015), to study how borrowing constraints would affect the accumulation of assets and market dynamics.

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Appendix A. Proofs

Proof of Proposition 1. To solve for the consumption weight of an individual i, take the market clearing condition in consumption and divide through by agent i's consumption, then substitute for individual consumption using Eq. (6)

$$\frac{1}{N}\sum_{j=1}^{N} c_{jt} = D_t \Leftrightarrow c_{it} = \frac{c_{it}}{\frac{1}{N}\sum_{j=1}^{N} c_{jt}} D_t = \left(\frac{N\left(e^{\rho t}\Lambda_i H_t\right)^{\frac{-1}{\gamma_i}}}{\sum_{j=1}^{N}\left(e^{\rho t}\Lambda_j H_t\right)^{\frac{-1}{\gamma_j}}} \right) D_t = \omega_{it} D_t$$

$\square$

Proof of Lemma 1. Model consumption as a geometric Brownian motion:

$$\frac{dc_{it}}{c_{it}} = \mu_{it}^{c} dt + \sigma_{it}^{c} dW_t \tag{22}$$

Solve for H_t in Eq. (6), apply Itô's lemma, and match coefficients to those in Eq. (5), which gives the result. $\square$

Proof of Proposition 2. Recall the definition of consumption dynamics in (22) and the market clearing condition for consumption in (4). Apply Itô's lemma to the market clearing condition:

$$\frac{1}{N}\sum_{i} c_{it} = D_t \Rightarrow \frac{1}{N}\sum_{i} dc_{it} = dD_t$$

By matching coefficients we find

$$\mu_D = \frac{1}{N}\sum_{i} \omega_{it}\mu_{it}^{c} \quad , \quad \sigma_D = \frac{1}{N}\sum_{i} \omega_{it}\sigma_{it}^{c}$$

Now use Lemma 1 to substitute the values for consumption drift and diffusion, then solve for the interest rate and the market price of risk, which gives the result. $\square$

Proof of Proposition 3. Assume that consumption weights follow a geometric

Brownian motion given by

$$\frac{d\omega_{it}}{\omega_{it}} = \mu^{\omega}_{it} dt + \sigma^{\omega}_{it} dW_t \tag{23}$$

Apply of Itô's lemma to the definition of consumption weights in (**??**):

$$\omega_{it} = \frac{\left(\Lambda^i e^{\rho t} H_t\right)^{\frac{-1}{\gamma_i}}}{\sum_j \left(\Lambda_j e^{\rho t} H_t\right)^{\frac{-1}{\gamma_j}}} \Leftrightarrow \omega^i(t) = \left[\sum_j \Lambda_j^{\frac{-1}{\gamma_j}} \Lambda_i^{\frac{1}{\gamma_i}} \left(e^{\rho t} H_t\right)^{\frac{1}{\gamma_i} - \frac{1}{\gamma_j}}\right]^{-1}$$

Matching coefficients gives the result. □

Proof of Propostition 4. Assume there exists a Markovian equilibrium in D_t. Then an individual's Hamilton-Jacobi-Bellman (HJB) equation writes

$$\begin{aligned} 0 = \max_{c_{it},\pi_{it}} \Bigg\{ & e^{-\rho t} \frac{c_{it}^{1-\gamma_i} - 1}{1-\gamma_i} + \frac{\partial J_{it}}{\partial t} + \left[X_{it}\left(r_t + \pi_{it}\sigma_t\theta_t\right) - c_{it}\right] \frac{\partial J_{it}}{\partial X_{it}} \\ & + \mu_D D_t \frac{\partial J_{it}}{\partial D_t} + \sigma_D \sigma_t \pi_{it} D_t X_{it} \frac{\partial^2 J_{it}}{\partial X_{it} \partial D_t} + \frac{1}{2}\left[X_{it}^2 \pi_{.it}^2 \sigma_t^2 \frac{\partial^2 J_{it}}{\partial X_{it}^2} + \sigma_D^2 D_t^2 \frac{\partial^2 J_{it}}{\partial D_t^2}\right] \Bigg\} \end{aligned} \tag{24}$$

subject to the transversality condition $\mathbb{E}_t J_{it} \to 0$ for all i s.t. $\gamma_i > \underline{\gamma}$, as the agent with the lowest risk aversion will dominate in the long run (Cvitanić et al. (2011)). First order conditions imply

$$c_{it} = \left(e^{-\rho t} \frac{\partial J_{it}}{\partial X_{it}}\right)^{\gamma_i} \tag{25}$$

$$\pi_{it} = -\left(X_{it}\sigma_t \frac{\partial^2 J_{it}}{\partial X_{it}^2}\right)^{-1} \left[\theta_t \frac{\partial J_{it}}{\partial X_{it}} + \sigma_D D_t \frac{\partial^2 J_{it}}{\partial X_{it}\partial D_t}\right] \tag{26}$$

Assume that the value function is separable as

$$J_{it}(X_{it}, D_t) = e^{-\rho t} \frac{X_{it}^{1-\gamma_i} V_i(D_t)^{\gamma_i}}{1-\gamma_i} \tag{27}$$

Substituting Eq. (27) into Eqs. (25) and (26) gives

$$c_{it} = \frac{X_{it}}{V_i(D_t)} \tag{28}$$

$$\pi_{it} = \frac{1}{\gamma_i \sigma_t} \left(\gamma_i \sigma_D D_t \frac{V_i'(D_t)}{V_i(D_t)} + \theta_t \right) \tag{29}$$

which shows that $V_i(D_t)$ is the wealth-consumption ratio as a function of the dividend. Next, substitute Eqs. (27) to (29) into Eq. (24) and simplify to find

$$\begin{aligned} 0 =& 1 + \frac{\sigma_D^2 D_t^2}{2} V_i''(D_t) + \left[\frac{1-\gamma_i}{\gamma_i} \theta_t \sigma_D + \mu_D \right] D_t V_i'(D_t) \\ &+ \left[(1-\gamma_i) r_t - \rho + \frac{1-\gamma_i}{2\gamma_i} \theta_t^2 \right] \frac{V_i(D_t)}{\gamma_i} \end{aligned} \tag{30}$$

Which gives an ode for the wealth-consumption ratio. On the boundaries $D = 0$ and $D \to \infty$, the most risk averse and the least risk averse agent dominates, respectively (Cvitanić et al. (2011)). The boundary conditions correspond to the value functions for individuals when prices are set by these dominant agents. One could identically take the limit in the ODE itself to arrive at the boundary conditions.

Define the price-dividend ratio as a function of the single state variable: $\mathcal{S}_t(D_t) = \frac{S_t}{D_t}$. Apply Itô's lemma to $D_t \mathcal{S}_t = S_t$ and match coefficients to find

$$\begin{aligned} \mu_t &= D_t^2 + \frac{(\sigma_D D_t)^2}{2} \frac{\mathcal{S}_t''(D_t)}{\mathcal{S}_t(D_t)} D_t + D_t \mu_D + \frac{\mathcal{S}_t'(D_t)}{\mathcal{S}_t(D_t)} (\sigma_D D_t)^2 \\ \sigma_t &= \sigma_D \left(1 + D_t \frac{\mathcal{S}_t'(D_t)}{\mathcal{S}_t(D_t)} \right) \end{aligned}$$

Taking the market clearing condition for wealth, rewrite $\mathcal{S}_t(D_t)$ as a function of D_t:

$$S_t = \frac{1}{N} \sum_i X_{it} \Leftrightarrow \frac{S_t}{D_t} = \mathcal{S}_t(D_t) = \frac{1}{N} \sum_i \frac{X_{it}}{D_t} = \frac{1}{N} \sum_i \frac{X_{it}}{c_{it}} \frac{c_{it}}{D_t} = \frac{1}{N} \sum_i V(D_t) \omega_{it}$$

which gives $\mathcal{S}_t(D_t)$ given that $\omega_{it} = f_i(D_t)$

□

Proof of Propostition 5. This proof proceeds identically to Chabakauri (2015). Let $V_i(D) \in C^1[0,\infty) \cup C^2[0,\infty)$, $0 < V_i(D) \leq C_1$, $|\pi_{it}\sigma_t| < C1$, and $|\theta_t| < C_1$, where C_1 is a constant. Additionally, assume

$$\mathbb{E}\int_0^\infty e^{-\rho t}\frac{c_{it}^{1-\gamma_i}}{1-\gamma_i}dt < \infty \tag{31}$$

$$\mathbb{E}\int_0^T J_i(X_{it}, D_t, t)^2 dt < \infty \qquad \forall\ T > 0 \tag{32}$$

$$\limsup_{T\to\infty} \mathbb{E}J_i(X_{it}, D_t, t) \geq 0 \tag{33}$$

Define $U_t = \int_0^t e^{-\rho t}c_{i\tau}^{1-\gamma_i}/(1-\gamma_i)d\tau + J_i(X_{it}, D_t, t)$, which satisfies $dU_t = \mu_{Ut}dt + \sigma_{Ut}dW_t$ such that

$$\mu_{Ut} = \left(e^{-\rho t}\frac{c_{it}^{1-\gamma_i} - 1}{1-\gamma_i} + \frac{\partial J_{it}}{\partial t} + \left[X_{it}\left(r_t + \pi_{it}\sigma_t\theta_t\right) - c_{it}\right]\frac{\partial J_{it}}{\partial X_{it}} + \mu_D D_t \frac{\partial J_{it}}{\partial D_t}\right. \tag{34}$$

$$\left. + \sigma_D\sigma_t\pi_{it}D_t X_{it}\frac{\partial^2 J_{it}}{\partial X_{it}\partial D_t} + \frac{1}{2}\left[X_{it}^2\pi_{.it}^2\sigma_t^2\frac{\partial^2 J_{it}}{\partial X_{it}^2} + \sigma_D^2 D_t^2\frac{\partial^2 J_{it}}{\partial D_t^2}\right]\right) \tag{35}$$

$$\sigma_{Ut} = J_{it}\left((1-\gamma_i)\pi_{it}\sigma_t + \gamma_i D_t\sigma_D\frac{V_i'(D)}{V_i(D)}\right) = J_{it}\left(\pi_{it}\sigma_t - \theta_t\right) \tag{36}$$

Notice that μ_{Ut} is simply the PDE inside the max operator in the HJB Eq. (24), thus $\mu_{Ut} \leq 0$. By the boundedness conditions, U_t is integrable and because its drift is negative it is a supermartingale, thus $U_t \geq \mathbb{E}_t U_T\ \forall\ t \leq T$, which is equivalent to

$$J_i(X_{it}, D_t, t) \geq \mathbb{E}_t\int_t^T e^{-\rho(\tau-t)}\frac{c_{i\tau}^{1-\gamma_t}}{1-\gamma_i}d\tau + \mathbb{E}_t J_i(X_{it}, D_t, T) \tag{37}$$

Since the first term is monotonic in T, by Eq. (33) and by the monotone convergence theorem we have

$$J_i(X_{it}, D_t, t) \geq \mathbb{E}_t\int_t^\infty e^{-\rho(\tau-t)}\frac{c_{i\tau}^{1-\gamma_t}}{1-\gamma_i}d\tau \tag{38}$$

Now to show the opposite, consider first the limit $\mathbb{E}_t J_i(X_{i\tau}, D_\tau, \tau) \to 0$ as

$\tau \to \infty$. Applying Itô's lemma to $J_i(X_{it}, D_t, t)$ and following similar steps as before, we find $dJ_{it} = J_{it}[\mu_{Jt}dt + \sigma_{Jt}dW_t]$ where

$$\mu_{Jt} = \frac{-1}{V_i(D)}$$
$$\sigma_{Jt} = \pi_{it}\sigma_t - \theta_t$$

by the first order conditions Eqs. (28) and (29). By the boundedness assumptions σ_{Jt} satisfies Novikov's conditions and we have that $d\eta_t = \eta_t\sigma_{Jt}dW_t$ acts as a change of measure to remove the Brownian term in J_{it}. We have

$$\begin{aligned}|\mathbb{E}_t J_i(X^*_{i\tau}, D_\tau, \tau)| &\leq \mathbb{E}_t\left[|J_{i\tau}|exp\left\{-\int_t^T \frac{1}{V_i(D_u)}du\right\}\frac{\eta_\tau}{\eta_t}\right] \\ &\leq |J_{it}|e^{-(T-t)/C_1}\mathbb{E}_t\frac{\eta_\tau}{\eta_t} = |J_{it}|e^{-(T-t)/C_1}\end{aligned}$$

Taking the limit in T gives the result.

Finally, define U^*_t as for U_t, except evaluated at the optimum consumption. Then

$$dU^*_t = J_{it}\left(\pi_{it}\sigma_t - \theta_t\right)dW_t \tag{39}$$

Again applying Novikov's condition we get that U^*_t is an exponential martingale, which gives

$$J_i(X_{it}, D_t, t) = \mathbb{E}_t\int_t^T e^{-\rho(\tau-t)}\frac{(c^*_{it})^{1-\gamma_t}}{1-\gamma_i}d\tau + \mathbb{E}_t J_i(X^*_{it}, D_t, T)$$

Finally, by the intermediate result the last term goes to zero, showing that we do indeed have the optimum. □

Appendix B. Numerical Methods

B.1. ODE Solution by Finite Difference

To solve for portfolios and wealth, one needs to solve the ode in Eq. (16). In this work I use finite difference methos (see Press (2007)). In the following I suppress the i subscript for clarity. Using a central difference scheme (assuming an evenly spaced grid), the ode for a given i can be approximated as

$$0 = 1 + a(D_k)\frac{V_{k+1} - 2V_k + V_{k-1}}{h^2} + b(D_k)\frac{V_{k+1} - V_{k-1}}{2h} + c(D_k)V_k$$

where D_k corresponds to the kth point in the grid, h the step size, $a(D_k) = \sigma_D^2 D_k^2/2$, $b(D_k) = ((1-\gamma_i)\theta(D_k)\sigma_D/\gamma_i + \mu_D)D_k$, $c(D_k) = ((1-\gamma_i)r(D_k) - \rho + (1-\gamma_i)\theta(D_k)^2/(2\gamma_i))/\gamma_i$. This can be rewritten as a system of linear equations:

$$0 = 1 + (x_k - y_k)V_{k-1} + (z_k - 2x_k)V_k + (x_k + y_k)V_{k+1}$$

where $x_k = a(D_k)/h^2$, $y_k = b(D_k)/2h$, and $z_k = c(D_k)$. Combining this system of equations with the boundary conditions in Eq. (17) one gets a system of $K-2$ equations in $K-2$ unknowns which takes a highly spars structure (Note: This paper takes the approach of fixing $lim_{D\to\infty}V'(D) = 0$, or a reflecting boundary condition. This seems to provide more stability and is confirmed by numerical simulations, although may not be the "best" approximation). This paper uses `scipy.sparse` to build the matrix equation and solve for the value functions.

Appendix C. Extension to Infinite Types

Consider the limiting case as $N \to \infty$. If we take the consumption weights, $\omega_{it} = \omega_t(\gamma_i, x_i)$, we have a function of an empirical mean, which converges to the mean with respect to the initial distribution by the Strong

Law of Large Numbers:

$$\omega_t(\gamma_i, x_i) = \frac{N\left(\Lambda(\gamma_i, x_i)e^{\rho t}H_t\right)^{\frac{-1}{\gamma_i}}}{\sum_{j=1}^{N}\left(\Lambda(\gamma_j, x_j)e^{\rho t}H_t\right)^{\frac{-1}{\gamma_j}}} \underset{N\to\infty}{\to} \frac{\left(\Lambda(\gamma, x)e^{\rho t}H_t\right)^{\frac{-1}{\gamma}}}{\int\left(\Lambda(\gamma, x)e^{\rho t}H_t\right)^{\frac{-1}{\gamma}} dG(\gamma, x)} = \omega_t(\gamma, x)$$

This result is logical when viewed through the lens of work on a continuum of agents à la Aumann (1964). However, the market clearing condition for consumption weights implies something intriguing about their relationship to the initial distribution. If we think of consumption weights as a ratio of probability measures, then they act as the Radon-Nikodym derivative of a stochastic measure with respect to the distribution of the initial condition. That is, define $\omega_t(\gamma, x) = \frac{dP_t(\gamma,x)}{dG(\gamma,x)}$. Then we have

$$\int \omega_t(\gamma, x)dG(\gamma, x) = \int \frac{dP_t(\gamma, x)}{dG(\gamma, x)}dG(\gamma, x) = \int dP_t(\gamma, x) = 1$$

The evolution of this distribution would be difficult to describe directly, but the expressions in Proposition 3 give the dynamics of this stochastic distribution. So $\omega_t(\gamma, x)$ allows one to calculate exactly the evolution of this stochastic distribution by use of a change of measure. Alternatively, one can think of $\omega_t(\gamma, x)$ as a sort of importance weight, where as the share of risky assets is concentrated towards one area in the support, the weight of this area grows in the determination of asset prices.

Additionally, the Radon-Nikodym interpretation allows one the accuracy of finite types as an approximation to continuous types. Say for instance we would like to discretize the above expression for the market clearing condition on $\omega_t(\gamma, x)$ using a Riemann sum with an evenly spaced partition (e.g. a midpoint rule):

$$\int \omega_t(\gamma, x)dG(\gamma, x) \approx \frac{(\overline{\gamma} - \underline{\gamma})(\overline{x} - \underline{x})}{JK} \sum_{k=1}^{K}\sum_{j=1}^{J} \omega_t(\gamma_k, x_j)g(\gamma_k, x_j) \qquad (40)$$

This looks quite similar to the market clearing conditions in the finite type model (Eq. (4)). Make the identification $N = JK$ and notice that since

$\omega_t(\gamma, x)$ is a geometric Brownian motion, $\omega_t(\gamma, x) = \omega_0(\gamma, x)\hat{\omega}_t(\gamma, x)$ where $\hat{\omega}_t(\gamma, x)$ is a stochastic process with initial value 1. If we define the initial condition on omega as $\omega_0(\gamma, x) = \frac{1}{(\overline{\gamma}-\underline{\gamma})(\overline{x}-\underline{x})g(\gamma,x)}$, then Eq. (40) becomes

$$\int \omega_t(\gamma, x) dG(\gamma, x) \approx \frac{1}{N} \sum_{k=1}^{K} \sum_{j=1}^{J} \hat{\omega}_t(\gamma_k, x_j) \tag{41}$$

Now Eq. (41) matches exactly the condition in Eq. (4). However, this equation has particular implications about the Radon-Nikodym derivative. From the definition of the Radon-Nikodym derivative we can write

$$P_t(A) = \int_A \omega_t(\gamma, x) dG(\gamma, x) = \int_A \omega_t(\gamma, x) g(\gamma, x) d\gamma dx$$

Substituting the imposed definition of $\omega_t(\gamma, x)$ we have

$$P_t(A) = \int_A \frac{\hat{\omega}_t(\gamma, x)}{(\overline{\gamma} - \underline{\gamma})(\overline{x} - \underline{x})} d\gamma dx$$

Now, since $\hat{\omega}_0(\gamma, x) = 1$, the above implies

$$P_0(A) = \int_A \frac{1}{(\overline{\gamma} - \underline{\gamma})(\overline{x} - \underline{x})} d\gamma dx = \int_A \omega_0(\gamma, x) g(\gamma, x) d\gamma dx$$

This implies that the evenly spaced grid approximation of preference distributions produces a particular assumption about the initial condition $\omega_0(\gamma, x)g(\gamma, x)$. The initial condition in such an approximation is limited to a uniform distribution such that $\omega_0(\gamma, x)g(\gamma, x) = \frac{1}{(\overline{\gamma}-\underline{\gamma})(\overline{x}-\underline{x})}$. Any change in initial consumption weights will change the underlying assumptions about the distribution $g(\gamma, x)$.

The continuous types model provides several useful modeling tools beyond finite types. First, the joint distribution of initial wealth and risk aversion is explicitly modeled. In a model of finite types over an evenly spaced grid one can only model a product distribution such that $\omega_0(\gamma, x)g(\gamma, x)$ is uniform. Second, but closely related, is the computational simplification provided by the continuum. One can simulate quadrature points to approximate a continuous distribution, whereas to do the same for the finite types model would require many simulated agents. Finally, the continuum allows one to coherently model the distribution of risk preferences if one believes there to be many preference types and if the number of types has an effect on model predictions.

Appendix D. Finite Types versus Continuous Types Simulation

For the continuous types model we must approximate the integrals in some way. For simplicity I use a trapezoidal rule. As an example, consider the definition of $\xi(t)$ and its associated quadrature approximation:

$$\xi_t = \int \frac{\omega(t,\gamma,x)}{\gamma} dG(\gamma,x) \approx \frac{\overline{\gamma}-\underline{\gamma}}{2(M-1)} \sum_{m=1}^{M-1} \left[\frac{\omega_t(\gamma_m)g(\gamma_m)}{\gamma_m} + \frac{\omega_t(\gamma_{m+1})g(\gamma_{m+1})}{\gamma_{m+1}} \right]$$

where (γ_m) is an evenly spaced grid. For finite types, changing the number of simulated points changes the distribution $g(\gamma)$ in the model, while for the continuous types solution, changing the number of quadrature points does not change the assumptions about $g(\gamma)$. This will be the key feature that differentiates the two solutions.

Consider the same uniform distribution as before. In this case the results look almost identical to the finite types case. In Figure 7 you can see that the interest rates are very similar. This similarity is driven by the integral approximations and the fact that we are using a uniform distribution over preferences. As pointed out in section C, we could *only* match a uniform initial distribution. To see this we can look at robustness results for a more complex assumption about the distribution of preferences and different methods of approximation.

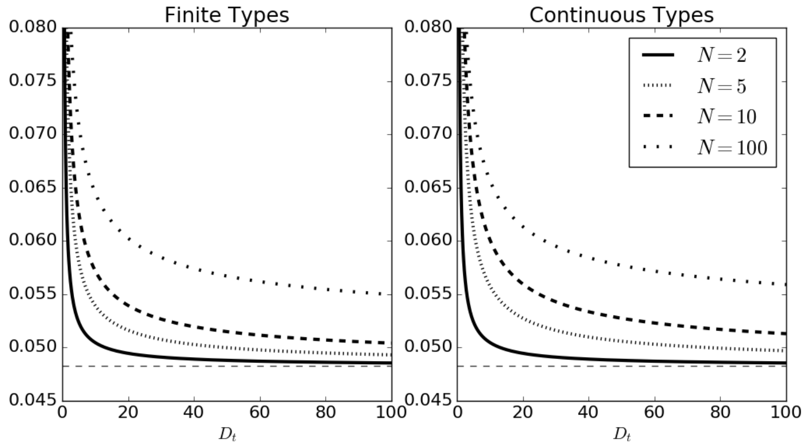

Fig. 7. Interest rates for different numbers of agents and quadrature points, respectively, under the assumption of a uniform distribution of preferences.

Appendix E. Simulation Robustness

Consider the case where preferences follow a Beta(2, 2) distribution over the same support. In this case, we could take several approaches to solving the finite types model. First, we could consider taking the same naive uniform approximation, fixing a uniform distribution of consumption weights over the same support. Second, we could consider initializing the consumption weights to match the same initial condition in the continuous types case, i.e. $\omega_0(\gamma) = g(\gamma)$. Third, we could attempt to use a monte-carlo approximation, drawing many agents from the distribution $g(\gamma)$. Finally, we could use the continuous types model, directly. The interest rate is presented in Figure 8 for each of these approximations.

You will notice that the solutions are substantially different. First, a uniform distribution of agents is a poor approximation to a non-uniform distribution of preferences (as one would expect). Second, the monte-carlo approximation converges to the continuous types solution very slowly, but is much better than the uniform approximation. Finally, one might think that changing the initial condition in $\omega_t(\gamma_i)$ to match the distribution of preferences one could recover the same solution. However, this simultaneously changes the assumptions about the distribution $g(\gamma)$, as was pointed out in

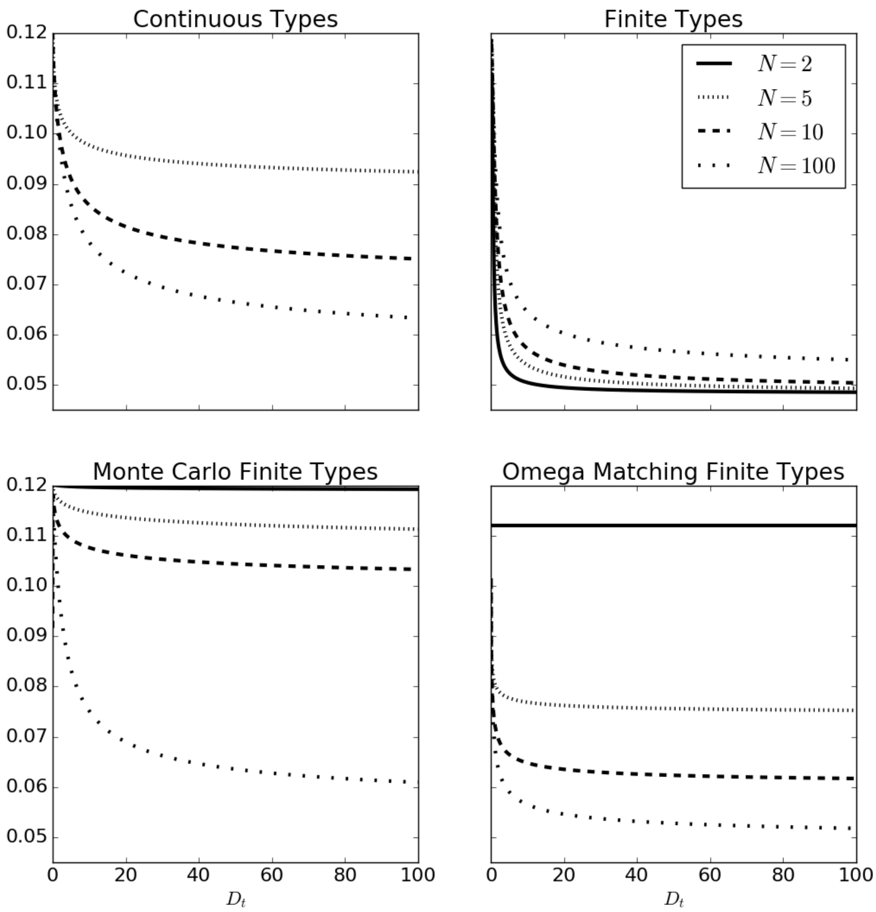

Fig. 8. Interest rate for different numbers of points and different approximation strategies assuming preferences distributed Beta(2, 2).

section C, producing a noticeably different interest rate.

In addition, the assumption of a Beta distribution over preferences changes the outcome for financial variables. You'll see in Figure 9 that not only is volatility substantially higher for a longer period of time, but leverage is as well. Additionally, both variables exhibit an inflection point. For very low values of D_t leverage cycles become pro-cyclical and the volatility smile inverts. This difference with the uniform distribution case is driven by the

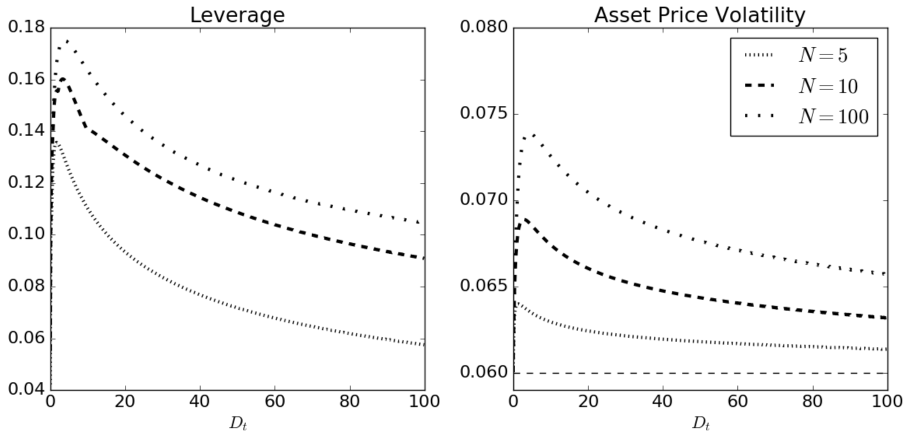

Fig. 9. Leverage and volatility for different numbers of quadrature points assuming continuous types distributed beta(2, 2).

volatility in borrowing. In order to produce a pro-cyclical leverage cycle, total borrowing must fall more quickly than total wealth. In this case, the risk neutral agent is constrained by their wealth being very low and thus limiting their borrowing. When dividends become low enough, their demand for borrowing collapses. In addition, pro-cyclical leverage cycles are accompanied by pro-cyclical volatility, or an inverted volatility skew. This comovement represents a testable implication.