Heterogeneous Risk Preferences in Financial Markets

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Abstract

In this paper I build a continuous time model of a complete financial market with N heterogeneous agents whose constant relative risk aversion (CRRA) preferences differ in their level of risk aversion. I find that preference heterogeneity is able to replicate a high market price of risk and a low risk-free rate by separating the markets for risky and risk free assets. This provides an explanation for the equity risk premium and risk-free rate puzzles, while avoiding a preference for early resolution of uncertainty inherent in non-seperable preferences, i.e. Epstein-Zin preferences. Additionally, I find that changing the number of preference types has a non-trivial effect on the solution. Finally, I show through a numerical example that the model predicts several phenomena observed in financial data, namely a correlation between dividend yields and the stochastic discount factor, a non-linear response of volatility to shocks, and both pro- and counter-cyclical leverage cycles depending on the assumptions about the distribution of preferences.

Introduction

Each day, trillions of dollars worth of financial assets change hands. This exchange must be driven by some dimension of investor heterogeneity, otherwise individuals would all hold the same portfolio. This paper focuses on heterogeneity in risk preferences as a driver of trade and considers how the degree of heterogeneity affects model predictions. In fact, it is shown how the introduction of more preference types allows the model to explain the equity risk premium puzzle of Mehra and Prescott (1985) and the risk free rate puzzle of Weil (1989). Although the average individual determines the market price

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of risk, the variance in preferences determines the risk free rate. These two moments of the distribution of preferences provide two parameters with which to match both the equity risk premium and the risk free rate. This implies that these two puzzles are simply artifacts of the representative agent assumption. The effect works through separating the markets for risky and risk free assets.

Markets for risky and risk free assets clear at different preference levels because these two assets provide two different services to the individual. Risky assets provide agents with claims on dividends, as well as capital gains from changes in prices. Risk free assets provide a way to hedge the co-movement between one's own consumption and returns on risky assets. In the lingo of the capital asset pricing model, the risk free asset allows one to choose a position along the security market line, increasing or reducing their exposure to fundamental volatility. The price of risk free assets depends on how much exposure individuals would like to have to this fundamental volatility. Agents borrow to increase their exposure if they have low risk aversion and lend if they have high risk aversion. The marginal agent in these two markets is not necessarily the same. Section 2.3 shows how the two markets overlap, producing several groups of agents who desire different portfolio compositions. The distribution of these groups determines the market price of risk and the risk free rate. The risk free rate is low and the market price of risk high when the variance in preferences is high. This is driven by demand and supply in the market for risk free assets. If the variance in the distribution of preferences is zero, there is no demand and no supply for risk free borrowing.

In addition to providing a partial explanation for the equity risk premium, results for two types are shown to not necessarily generalize to many types. Changing the number of preference types substantially alters model predictions. As previously mentioned, two moments of the distribution of preferences are important for determining the market price of risk and risk free rate. Although it is possible to construct models with different numbers of types which match one of these moments, it is not possible to add a single agent to a model and match both moments simultaneously. For a practical example, consider a sample of 2 random integers and attempt to choose a third integer so that the new set of 3 integers has the same mean and variance as the first set. Given two equations in one unknown, there are not enough free variables. In the same way, the addition of more agents will alter either the mean or the variance (or both) in the distribution of preferences. This change will alter the level and dynamics of investment opportunities in every period of the model.

In the equity risk premium puzzle first put forth by Mehra and Prescott (1985), under a representative agent with CRRA preferences a single parameter determines both the market price of risk and the risk free rate. One needs a very high level of risk aversion to match excess returns on equity, but this produces a high risk free rate. Weil (1989) pointed out that if one attempts to solve this issue by introducing a second preference parameter via Epstein-Zin preferences, the representative agent must have a very high elasticity of inter-temporal substitution, a phenomenon known as the risk-free rate puzzle. In addition, Epstein et al. (2014) recently added to the controversy by showing that an Epstein-Zin representative agent must be willing to pay a very large amount for the early resolution of uncertainty, implying that solving one puzzle with these preferences creates another. In the model presented here, agents have constant relative risk aversion (a fact noted empirically in Brunnermeier and Nagel (2008)) and are thus indifferent to the resolution of uncertainty. Despite this, the model is still able to produce a high equity risk premium and low risk-free rate by separating the markets for risky and risk-free assets.

Many authors have studied the problem of heterogeneous preferences under two preference types (e.g. Dumas (1989); Coen-Pirani (2004); Guvenen (2006); Bhamra and Uppal (2014); Chabakauri (2013, 2015); Gârleanu and Panageas (2015); Cozzi (2011)). However, there are few articles which study many types. Cvitanić et al. (2011) study an economy populated by N agents who differ in their risk aversion parameter, their rate of time preference, and their beliefs. Those authors focus on issues of long run survival, while the present paper studies how changes in the distribution of preferences affect the short run dynamics of the model, focusing on a single aspect of heterogeneity.[1] Finally, a Markovian equilibrium in a single state variable is characterized. This allows one to study how financial variables evolve over the entire state space as opposed to the Malliavan calculus characterization in Cvitanić et al. (2011).

The model also relates to the study of games with a large number of heterogeneous agents, otherwise known as Mean Field Games. Games featuring a continuum of agents hearken back to Aumann (1964). However, their study in a stochastic setting has recently garnered a large amount of attention thanks to a series of papers by Jean-Michel Lasry and Pierre-Louis Lions (Lasry and Lions (2006a), Lasry and Lions (2006b), Lasry and Lions (2007)). These authors studied the limit of N-player stochastic differential games as $N \to \infty$ and agents' risk is idiosyncratic, dubbing the system of equations governing the limit a "Mean Field Game" (MFG). Their work has then been applied to macroeconomics in works such as Moll (2014), Achdou et al. (2014), and Kaplan et al. (2016). However, these papers focus on idiosyncratic risk and do not study the problem of aggregate shocks, which is a lively area of research (e.g. Carmona et al. (2014), Carmona and Delarue (2013), Chassagneux et al. (2014), and Cardaliaguet et al. (2015)) and which encompass classic macroeconomic models such as Krusell and Smith (1998). The approach is either to use a stochastic Pontryagin maximum principle to derive a system of forward-backward stochastic differential equations governing the solution, or to define an infinite dimensional PDE governing the agents' value functions. These approaches are clear from a mathematical perspective, but very difficult to formulate for more complex economic models (although Ahn et al. (2016) provides a method for approximating the

[1] Additionally, in the appendix I characterize the limit as $N \to \infty$.

solution using the latter method).

This paper takes a different approach, solving the model with common noise using Girsanov theory in the style of Harrison and Pliska (1981) and Karatzas et al. (1987). One then recognizes that the SDF can be written as a function of a single state variable. This implies a Markovian equilibrium in the style of Chabakauri (2013) and Chabakauri (2015). The solution is characterized by mean field dependence through the control, as opposed to the state. This points towards a new way to consider control in mean field financial models. If the dynamics of the stochastic discount factor can be written as a function of a small number of state variables, then the atomistic agents do not need to consider the entire distribution of individual states in order to solve their problem. This result is driven by complete markets, as ratios of marginal utilities are constant, and requires further study for incomplete markets.

The paper is organized as follows: in Section 1, I construct a continuous time model of financial markets populated by a finite number of agents who differ in their preferences towards risk. Section 2 characterizes the equilibrium. Section 3 provides numerical results. Section 4 concludes. The more technical analysis and proofs have been relegated to the appendix.

1. The Model

Consider a continuous time Lucas (1978) economy populated by a number, N, of heterogeneous agents indexed by $i \in \{1, 2, ..., N\}$. All agents discount future utility at the same exponential rate ρ. Each agent has instantaneous preferences over consumption given by constant relative risk aversion (CRRA) utility with relative risk aversion γ_i:

$$U_i(c_{it}) = \frac{c_{it}^{1-\gamma_i}}{1-\gamma_i} \qquad\qquad \forall i \in \{1, 2, ..., N\}$$

Agents can continuously trade in shares, α_{it}, of a per-capita dividend process, D_t, which follows a geometric Brownian motion:

$$\frac{dD_t}{D_t} = \mu_D dt + \sigma_D dW_t \tag{1}$$

where μ_D and σ_D are constants, and where D_0 is given. In addition agents can trade in an instantaneously risk-free bond, whose shares are denoted b_{it}. Initial shares in the risky asset are drawn according to a density $(\gamma_i, \alpha_{i0}) \sim g(\gamma, \alpha)$ and initial bond holdings are assumed to be zero. Risky share prices, S_t, and risk-free bond prices, S_t^0, follow an Itô process and an exponential process, respectively:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t \qquad (2) \qquad\qquad \frac{dS_t^0}{S_t^0} = r_t dt \qquad (3)$$

These assumptions imply that an agent's wealth is defined by $X_{it} = \alpha_{it} S_0 + b_{it} S_t^0$. Denote by $\pi_{it} = \alpha_{it} S_t / X_{it}$ the share of an individual's wealth invested in the risky stock.

1.1. Budget Constraints and Individual Optimization

An individual agent's constrained maximization subject to instantaneous changes in wealth can be written as:

$$\begin{aligned}
\max_{\{c_{it},\pi_{it}\}_{t=0}^{\infty}} \quad & \mathbb{E}\int_0^\infty e^{-\rho t}\frac{c_{it}^{1-\gamma_i}}{1-\gamma_i}dt \\
\text{s.t.} \quad & dX_{it} = \left[X_{it}\left(r_t + \pi_{it}\left(\mu_t + \frac{D_t}{S_t} - r_t\right)\right) - c_{it}\right]dt \\
& + \pi_{it} X_{it}\sigma_t dW_t
\end{aligned}$$

where the constraint represents the dynamic budget of an individual.

1.2. Market Clearing

Markets for consumption, wealth, and risk-free borrowing are assumed to clear, such that

$$\frac{1}{N}\sum_i c_{it} = D_t \ , \ \frac{1}{N}\sum_i (1-\pi_{it})X_{it} = 0 \ , \ \frac{1}{N}\sum_i \pi_{it} X_{it} = S_t \qquad (4)$$

1.3. Equilibrium

Definition 1. *An equilibrium in this economy is defined by a set of processes* $\{r_t, S_t, \{c_{it}, X_{it}, \pi_{it}\}_{i=1}^N\}\ \forall\ t$, *given preferences and endowments, such that* $\{c_{it}, X_{it}, \pi_{it}\}$ *solve the agents' individual optimization problems and the market clearing conditions in Eq.* (4) *are satisfied.*

I consider Markovian equilibria where the problem can be written as a function of some finite number of state variables. In particular, it will be shown that D_t is a sufficient state variable to characterize the equilibrium.

2. Equilibrium Characterization

To solve this problem I first use the martingale method to show how the SDF can be written as a function of a single state variable. I then use the Hamilton-Jacobi-Bellman (HJB) equation to derive a system of ordinary differential equations which determines the portfolio, stock price, and stock price volatility.

2.1. *The Static Problem*

Following Karatzas and Shreve (1998) define the SDF, H_t, as

$$\frac{dH_t}{H_t} = -r_t dt - \theta_t dW_t \qquad \textit{where} \qquad \theta_t = \frac{\mu_t + \frac{D_t}{S_t} - r_t}{\sigma_t} \tag{5}$$

where $H_0 = 1.0$. Here θ_t represents the market price of risk. Following Proposition 2.6 from Karatzas et al. (1987), we can rewrite each agent's dynamic problem as a static one beginning at time $t = 0$

$$\begin{aligned} \max_{\{c_{it}\}_{t=0}^{\infty}} \quad & \mathbb{E} \int_0^{\infty} e^{-\rho t} \frac{c_{it}^{1-\gamma_i}}{1-\gamma_i} dt \\ \text{s.t.} \quad & \mathbb{E} \int_0^{\infty} H_t c_{it} dt \leq X_{i0} \end{aligned}$$

If we denote by Λ_i the Lagrange multiplier in individual i's problem, then the first order conditions can be rewritten as

$$c_{it} = \left(e^{\rho t} \Lambda_i H_t\right)^{\frac{-1}{\gamma_i}} \tag{6}$$

which holds for every agent in every period.

Given each agent's first order conditions, we can derive an expression for consumption as a fraction of per-capita dividends.

Lemma 1. *One can define the consumption of individual, i, at any time, t, as a share ω_{it} of the per-capita dividend, D_t, such that*

$$c_{it} = \omega_{it} D_t \quad \textit{where} \quad \omega_{it} = \frac{N \left(\Lambda_i e^{\rho t} H_t\right)^{\frac{-1}{\gamma_i}}}{\sum_j \left(\Lambda_j e^{\rho t} H_t\right)^{\frac{-1}{\gamma_j}}} \tag{7}$$

This expression recalls the results in Basak and Cuoco (1998) or Cuoco and He (1994), where ω_{it} acts like a time-varying Pareto-Negishi weight. In those works, however, this weight arises from an imperfection in the information structure or some exogenous constraint. Here the markets are complete, but this weight is still time varying. The weight an agent gives to the SDF differs depending on the agents' risk aversion, despite the value of the SDF being equal across agents. These consumption weights represent a driving quantity in the model and can be defined implicitly in terms of aggregate consumption, as in the classical results of Arrow and Debreu (1954).

Lemma 2. *Consumption weights can be defined as an implicit function such that* $\omega_{it} = \omega_i(D_t)$*, which satisfies for all* i

$$\frac{1}{N}\sum_j \lambda_{ji}^{\frac{-1}{\gamma_j}}\,\omega_i(D_t)^{\frac{\gamma_i}{\gamma_j}}\,D_t^{\frac{\gamma_j-\gamma_i}{\gamma_j}} = 1 \;\; \textit{where}\;\; \lambda_{ji} = \frac{\Lambda_j}{\Lambda_i} \tag{8}$$

The ability to define the consumption weight as an implicit function of the dividend provides a convenient tool for the study of the model's equilibrium. It is well understood (Cvitanić et al. (2011)) that the dominant agent in the long run depends on the long run level of aggregate consumption. If $D_t \to \infty$, then the least risk averse agent will accumulate all of the consumption weight, while if $D_t \to 0$ then the most risk averse agent will dominate. However, Lemma 2 characterizes consumption weights over the entire state space. In addition it can be shown that consumption weights are increasing/decreasing as a function of D_t for the least/most risk averse agent, respectively.

2.2. The Risk-Free Rate and Market Price of Risk

It is possible to define the interest rate and market price of risk as functions of the consumption weights. This definition in turn implies they can be calculated as functions of a single state, given the characterization of consumption weights over D_t. The following lemma is useful to derive an expression for the risk free rate and the market price of risk and sheds light on individual consumption patterns:

Lemma 3. *An agent's consumption follows an Itô process with drift and diffusion coefficients* $c_{it}\mu_{it}^c$ *and* $c_{it}\sigma_{it}^c$ *such that*

$$r_t = \rho + \mu_{it}^c\gamma_i - (1+\gamma_i)\gamma_i\frac{(\sigma_{it}^c)^2}{2}$$
$$\theta_t = \sigma_{it}^c\gamma_i$$

which hold simultaneously for all i*.*

These formulas resemble those one would find in a standard representative agent model. However, these expressions hold simultaneously for all agents. Shocks cause the growth rate and volatility of consumption for each agent to adjust, while for a representative agent they would be replaced by the drift and diffusion of the dividend process. We can rewrite Lemma 3 in terms of μ_{it}^c and σ_{it}^c and differentiate in order to better understand how these values adjust:

$$\frac{\partial \mu_{it}^c}{\partial \theta_t} = \frac{1+\gamma_i}{\gamma_i^2}\theta_t \quad (9) \qquad \frac{\partial \sigma_{it}^c}{\partial \theta_t} = \frac{1}{\gamma_i} \quad (11)$$

$$\frac{\partial \mu_{it}^c}{\partial r_t} = \frac{1}{\gamma_i} \quad (10) \qquad \frac{\partial \sigma_{it}^c}{\partial r_t} = 0 \quad (12)$$

The derivatives of individual consumption parameters imply heterogeneity in the size of response to changes in financial variables, but homogeneity in sign. Eqs. (9) and (10) imply that the growth rate of every individual's consumption is increasing in both the market price of risk and in the interest rate. A higher market price of risk implies greater returns. These returns mean any given agent earns more on their portfolio and a wealth effect dominates. In regards to interest rates, agents choose their consumption growth conditional on their consumption at time t. Thus any change in the interest rate must pivot their life-time budget constraint through this point. This produces a positive and dominant wealth effect whether an agent is a net-borrower or -lender, as an increase in the interest rate makes lifetime consumption less costly. In addition, an increase in the market price of risk increases volatility of consumption for all agents, while the interest rate has no effect (Eqs. (11) and (12)). A higher market price of risk implies a greater volatility (in absolute value) of the SDF. Because of this, the present value of discounted future consumption becomes more volatile and in turn consumption becomes more volatile.

Given Lemma 3, we can derive expressions for the market price of risk and the risk free rate:

Proposition 1. *The interest rate and market price of risk are fully determined by the sufficient statistics* $\xi_t = \frac{1}{N}\sum_i \frac{\omega_{it}}{\gamma_i}$ *and* $\phi_t = \frac{1}{N}\sum_i \frac{\omega_{it}}{\gamma_i^2}$ *such that*

$$r_t = \rho + \frac{\mu_D}{\xi_t} - \frac{1}{2}\frac{\xi_t + \phi_t}{\xi_t^3}\sigma_D^2 \quad (13)$$

$$\theta_t = \frac{\sigma_D}{\xi_t} \quad (14)$$

Proposition 1 is in terms of only certain moments of the joint distribution of consumption shares and risk aversion: $\xi_t = \frac{1}{N}\sum_i \frac{\omega_{it}}{\gamma_i}$ and $\phi_t = \frac{1}{N}\sum_i \frac{\omega_{it}}{\gamma_i^2}$. These moments represent weighted averages of elasticity of intertemporal substitution (EIS). An agent's preferences only affect the market clearing interest rate and market price of risk up to their share in consumption.

The expressions in Proposition 1 are similar to those one would find in a representative agent economy. The market price of risk is instantaneously equal to that which would prevail in a representative agent economy populated by an agent whose elasticity of intertemporal substitution is equal to ξ_t. The interest rate is reminiscent of the interest rate in the same hypothetical economy, but not quite equal. We could rewrite Eq. (13) as the interest rate that would prevail in our hypothetical economy plus an extra term:

$$r_t = \rho + \frac{\mu_D}{\xi_t} - \frac{1}{2}\frac{\xi_t + 1}{\xi_t^2}\sigma_D^2 - \mathbf{\frac{1}{2}\frac{1}{\xi_t}\left(\frac{\phi_t}{\xi_t^2} - 1\right)\sigma_D^2}$$

This additional term (in bold) would be zero if $\phi_t = \xi_t^2$. However, we can apply the discrete version of Jensen's inequality to show that $\phi_t > \xi_t^2, \quad \forall t < \infty$ if $N > 1$. This causes the additional term to be strictly negative. The risk free rate is then lower than it would be in an economy populated by a representative agent. This introduces a sort of "heterogeneity wedge", which I'll define as $\frac{\phi_t}{\xi_t^2} > 1$, between the price of risk and the price for risk free borrowing. This wedge is also equal to one plus the squared coefficient of variation of the weighted EIS. The wedge will be higher when the variation in the EIS is higher, weighted by consumption shares. The driving force behind the heterogeneity wedge is the separation between the marginal agents in the markets for risky and risk-free assets that occurs when agents differ in their preferences towards risk.

2.3. Equity Risk Premium and Marginal Agents

This subsection discusses a metaphorical comparison to partial equilibrium to better explain the intuition for why the market price of risk can be high and interest rate low when preferences are heterogeneous[2]. The equity risk premium and risk free rate puzzles focus on the problem of the representative agent model producing both a high risk premium and high risk free rate or requiring a very impatient agent, respectively, in order to match the equity risk premium and risk free rate. Heterogeneous preferences can match both of these financial variables by separating the markets for risky and risk free assets.

The way in which preference heterogeneity generates a low risk free rate and high market price of risk can be seen through the marginal agents in the markets for risky and risk free assets. Define $\{\gamma_{rt}, \gamma_{\theta t}\}$ to be the RRA parameters in a representative agent economy that would produce the same interest rate and market price of risk, respectively, as in the heterogeneous preferences economy. These preference levels can be interpreted as the marginal agent in each market and are given in Proposition 2

Proposition 2. *The marginal preference levels γ_{rt} and $\gamma_{\theta t}$ are given by*

$$\gamma_{rt} = \frac{\mu_D}{\sigma_D^2} - \frac{1}{2} \pm \sqrt{\frac{\mu_D}{\sigma_D^2}\left(\frac{\mu_D}{\sigma_D^2} - 1 - \frac{2}{\xi_t}\right) + \frac{\xi_t + \phi_t}{\xi_t^3} + \frac{1}{4}}$$

$$\gamma_{\theta t} = \frac{1}{\xi_t}$$

[2] The marginal agents discussed in this section may or may not exist in the economy, but are useful for discussing individuals relative to market clearing preference levels.

It cannot be ruled out that the two roots $\gamma_{rt\pm}$ exist, but it can be shown that

$$\underline{\gamma} \leq \gamma_{rt-} \leq \gamma_{\theta t} \leq \gamma_{rt+} \leq \overline{\gamma}$$

The inequalities in Proposition 2 imply that the markets for risky and risk-free assets separate and do not coincide in finite t. This separation creates four groups of agents: investors, divestors, borrowers, and lenders. In addition, these groups overlap and individuals may desire to borrow or lend at the same time that they wish to invest or divest (see Figure 1). Each market can be thought of as an auction. When a single representative agent bids, their price must clear both markets. However, when many heterogeneous agents show up to each market there is nothing to impose a single market clearing preference level. In fact, the non-linearity/non-monotonicity of the interest rate as a function of relative risk aversion ensures that the two preference levels do not coincide in finite time.

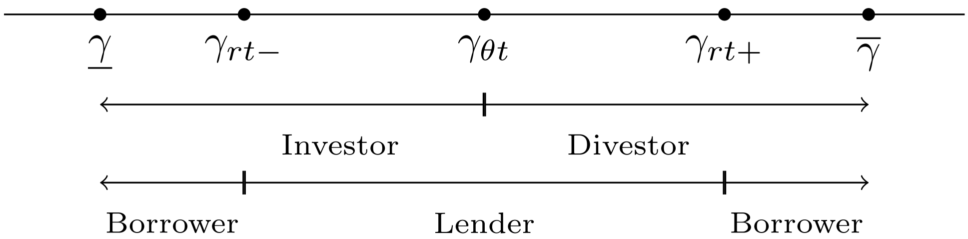
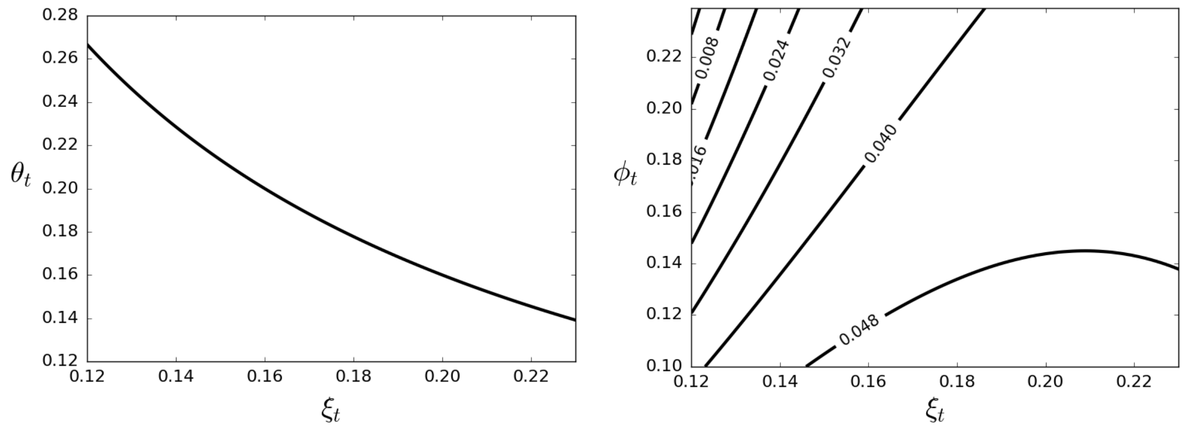


Fig. 1. An agent's position relative to γ_{rt} and $\gamma_{\theta t}$ sheds light on their consumption and saving decisions.

The overlap between these two markets causes the interest rate to be low relative to the market price of risk, producing a high equity risk premium, which can explain the equity risk premium and risk-free rate puzzles of Mehra and Prescott (1985) and Weil (1989). Figure 2 plots the market price of risk and risk free rate for different values of ξ_t and ϕ_t. When ξ_t is low, the market price of risk is high. This corresponds to the marginal preference level γ_{θ_t} being high, or to a very risk averse agent pricing the risky asset. At the same time a high ϕ_t produces a low risk free rate. This corresponds to a high variance in preferences, or a greater diversity in the market. In this case γ_{rt-} is very low and the marginal agent pricing risk free assets has low risk aversion. This can be explained through a supply and demand argument, as the variance in the distribution of preferences determines both the supply and demand for bonds. Increasing the variance while keeping the mean of the distribution fixed requires shifting consumption weights towards the high risk aversion agents, as their EIS is very low. Risk averse agents tend to be net lenders, as indicated by Figure 1, so this higher variance shifts out the demand for bonds and shifts in the supply. This causes an increase in the price of bonds and, in turn, a fall in the interest rate. Heterogeneity in preferences can in this way produce a low interest rate and a high market price of risk depending on where consumption weights are concentrated.



(a) Market Price of Risk

(b) Risk Free Rate

Fig. 2. Figure 2(a) plots the market price of risk as a function of average EIS, ξ_t. Figure 2(b) is a contour plot of the risk free rate as a function of average EIS and its square, ϕ_t (a measure of variance in preferences).

Preference heterogeneity beyond two preference types allows one the additional free variable necessary to match both the risk free rate and market price of risk, without introducing a preference for early resolution of uncertainty. In Epstein et al. (2014), it is shown that in order to match the equity risk premium using Epstein-Zin preferences, one must introduce a strong preference for the early resolution of uncertainty. In fact, the authors show that the representative agent would have to be willing to pay an exorbitant amount to resolve the uncertainty. In the model presented here, agents are CRRA and, thus, indifferent to early or late resolution of uncertainty. High expected returns in this model are driven in part by agents preferences, and in part by the dynamics of their consumption weights which can be derived explicitly.

2.4. Consumption Weight Dynamics

We can study the dynamics of an agent's consumption weight by applying Itô's lemma to the expression given in Lemma 1.

Proposition 3. *Assuming consumption weights follow an Itô process such that*

$$\frac{d\omega_{it}}{\omega_{it}} = \mu^{\omega}_{it}dt + \sigma^{\omega}_{it}dW_t$$

an application of Itô's lemma to (7) *gives expressions for* μ_{it}^{ω} *and* σ_{it}^{ω}*:*

$$\begin{aligned} \mu_{it}^{\omega} &= (r_t - \rho)\left(\frac{1}{\gamma_i} - \xi_t\right) && (15) \\ &\quad + \frac{\theta_t^2}{2}\left[\left(\frac{1}{\gamma_i^2} - \phi_t\right) - 2\xi_t\left(\frac{1}{\gamma_i} - \xi_t\right) + \left(\frac{1}{\gamma_i} - \xi_t\right)\right] \\ \sigma_{it}^{\omega} &= \theta_t\left(\frac{1}{\gamma_i} - \xi_t\right) && (16) \end{aligned}$$

Individual consumption weights evolve as functions of ξ_t and ϕ_t. Consider first the case where an agent's preferences coincide with the weighted average, ie $\gamma_i = \gamma_{\theta t} = \frac{1}{\xi_t}$ (as in section 2.3). In this case ω_{it} is instantaneously deterministic, i.e. $\sigma_{it}^{\omega} = 0$. This determinism arises because the agent represents the marginal agent in the market for risky assets. However, notice that in this case $\mu_{it}^{\omega} = \theta_t^2\left(\frac{1}{\gamma_{\theta t}^2} - \phi_t\right) = \sigma_D^2\left(1 - \frac{\phi_t}{\xi_t^2}\right)$. The agent is moving deterministically out of the marginal position. The speed with which this is occurring is driven by the heterogeneity wedge, $\frac{\phi_t}{\xi_t^2}$. When this wedge is high, the rate at which the marginal agent moves out of the marginal position is greater.

Next consider the case where an agent is more risk averse, that is $\gamma^i > \gamma_{\theta t}$. Then $\sigma_{it}^{\omega} < 0$ and agent i's weight is negatively correlated to the market. This negative correlation implies that if an agent is more risk averse than the average then their consumption share increases under negative shocks and decreases under positive shocks. This co-movement can be thought of as playing a "buy low, sell high" strategy for consumption. Conversely, less risk averse agents' shares co-vary positively with the market, i.e. $\sigma_{it}^{\omega} > 0$. These less risk averse agents are impatient and value present dividends over future dividends. A fall in the level makes agents poorer today and in the near future, so for these impatient agents the income effect dominates and their lifetime income shifts substantially. To compensate they must shift consumption. These are the day-traders, riding booms and busts to try to make a quick buck while not losing their shirts. Although they may benefit in the short run, their consumption share can be more volatile than the economy.

2.5. Asset Prices and Portfolios

Asset prices and portfolios can be derived via a combination of the HJB and the martingale method. In fact it can be shown (and verified) that the individual maximization problem can be formulated as a function of only two state variables, individual wealth and the dividend process. Recall the first order condition Eq. (6) and substitute this into the market clearing condition for consumption:

$$\frac{1}{N}\sum_i \left(e^{\rho t}H_t\right)^{\frac{-1}{\gamma_i}} = D_t$$

This shows that the SDF is an implicit function only of the dividend process D_t and time. Given this, it is natural to look for a solution to an agent's maximization in the two state variables (X_{it}, D_t) and time. However, given the homogeneity of CRRA preferences, the value functions factor and the dependence on wealth and time disappears, giving a solution in a single state variable. This solution is given in Proposition 4:

Proposition 4. *Given Lemma 2 and Proposition 1, it is possible to define the interest rate and market price of risk as functions of dividends, D_t, such that $r_t = r(D_t)$ and $\theta_t = \theta(D_t)$. Assuming there exists a Markovian equilibrium in D_t, the individuals' wealth-consumption ratios, $V_i(D_t) = X_{it}/c_{it}$, satisfy ODE's given by*

$$\begin{aligned} &\frac{\sigma_D^2 D_t^2}{2} V_i''(D_t) + \left[\frac{1-\gamma_i}{\gamma_i}\theta(D_t)\sigma_D + \mu_D\right] D_t V_i'(D_t) \\ &+ \left[(1-\gamma_i) r(D_t) - \rho + \frac{1-\gamma_i}{2\gamma_i}\theta(D_t)^2\right] \frac{V_i(D_t)}{\gamma_i} + 1 = 0 \end{aligned} \tag{17}$$

which satisfy boundary conditions

$$V_i(D^*) = \frac{\gamma_i}{\rho - (1-\gamma_i)\left(\frac{\theta(D^*)^2}{2\gamma_i} + r(D^*)\right)} \quad \textit{for } D^* \in \{0, \infty\} \tag{18}$$

This low dimensional, Markovian equilibrium provides a convenient tool for studying the solution to the model. Typically in heterogeneous preference models the number of state variables grows with the number of agents (see e.g. Gârleanu and Panageas (2015)). However, here the number of state variables remains small for an arbitrary number of agents. In addition, this solution provides a convenient characterization of portfolios and volatility as functions of individual wealth/consumption ratios.

Proposition 5. *If wealth/consumption ratios satisfy the ODE's given in Proposition 4, then portfolios and volatility are given by:*

$$\pi_i(D_t) = \frac{1}{\gamma_i \sigma_t}\left(\theta(D_t) + \gamma_i \sigma_D D_t \frac{V_i'(D_t)}{V_i(D_t)}\right) \tag{19}$$

$$\sigma(D_t) = \sigma_D\left(1 + D_t \frac{\mathcal{S}'(D_t)}{\mathcal{S}(D_t)}\right) \tag{20}$$

where the price/dividend ratio $\mathcal{S}(D_t)$ satisfies

$$\mathcal{S}(D_t) = \frac{1}{N}\sum_i \omega_{it} V_i(D_t) \tag{21}$$

Several facts jump out from these expressions. First, one can notice in Eq. (19) that portfolios separate into two terms as is common in the literature (see Merton (1969, 1971)): a myopic term and a hedging term. The myopic term is the market price of risk scaled by risk preferences and volatilities, representing the portfolio desired at the current point in time. The hedging term is determined by the shape of the agent's wealth consumption ratio and the fact that the investment opportunity set is inversely correlated with the market. A fall in the dividend process causes the more risk averse agents to dominate, reducing asset prices relative to dividends and increasing the dividend yield. Thus, the agent hedges possible losses in dividends with capital gains through their risky shares.

In addition, the model exhibits excess volatility which is inversely correlated to the market, determined by the shape of the price/dividend ratio. In Eq. (20) stock price volatility is the fundamental volatility, σ_D, plus the instantaneous volatility of the dividend scaled by the rate of change in the price/dividend ratio. As before, a fall in the dividend causes the risk averse agents to dominate, which in turn implies a fall in the price/dividend ratio. Thus the price/dividend ratio has a positive slope everywhere, but it is in fact concave. A negative shock causes this slope to rise, increasing the volatility. This fact is reminiscent of the implied volatility smile inherent in options pricing data (Fouque et al. (2011)).

Given the above proposed equilibrium, following identical steps as Chabakauri (2015), one can show existence of an equilibrium satisfied by the above proposition.

Proposition 6. *Suppose there exist bounded positive functions* $V(D) \in C^1[0,\infty) \cup C^2[0,\infty)$ *that satisfy the system of ODE's in Eq.* (17) *and boundary conditions in Eq.* (18). *Additionally, assume the processes* θ_t, σ_t, *and* π_{it} *are bounded and that* $|\sigma_t| > 0$. *Then there exists a Markovian equilibrium satisfied by Propostions 1, 4 and 5*

These equations recall the results in Chabakauri (2013, 2015), but characterize the equilibrium in a slightly different way. In particular the dimension remains low by studying an equilibrium taking only the dividend as a state variable instead of consumption weights. In fact, it is easy to study the limiting case of a continuum of types given that the dimension of the state space does not change with the number of preference types, although this has been left to the appendix.

3. Simulation Results and Analysis

In this section, I review the simulation strategy as well as some simulation results. The underlying assumption is that I am attempting to approximate a continuous distribution of types (for a recent survey on estimating risk preferences see Barseghyan et al. (2015) and for evidence on heterogeneity see Chiappori et al. (2012)). One could argue that this approximation is the goal of any model of heterogeneous risk preferences with finite types, as in Dumas (1989), Gârleanu and Panageas (2015), Chabakauri (2013), or Chabakauri (2015), but that one assumes finite types for tractability and with the hope that the results generalize. This section tries to convince you that results for two types do not necessarily generalize to more types. [3]

For all of the simulations, I hold the following group of parameters fixed at the given values: $\mu_D = 0.01$, $\sigma_D = 0.032$, and $\rho = 0.02$. These settings correspond to a yearly parameterization. Solutions are represented over the state space, $D_t \in [0, \infty)$, which is truncated for clarity as most of the action is in the lower regions. Additionally, assume that all agents begin with an initial wealth such that their consumption weights are equalized at $t = 0$. I present here results for different numbers of types and leave to the appendix continuous types and robustness results.

Assume preferences are distributed uniformly over $[1.1, 18.0]$. Consider changing the number of evenly spaced types over the support. I consider the cases where $N = 2, 5, 10$ and 100. Plots are presented for particular variables showing the results for the different settings. The change in the number of types is shown to have a significant effect on the level of all aggregate variables. This effect implies that mis-specification of the support of the distribution of risk preferences has a non-trivial effect on the model's short run predictions about market variables.

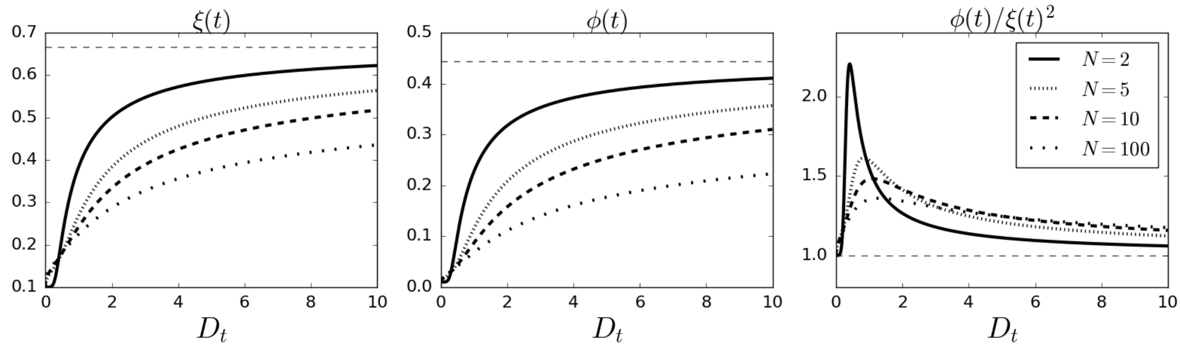


Fig. 3. Sufficient statistics for the distribution of the risk aversion with finite types, where $\xi_t = \frac{1}{N}\sum_{i=1}^{N}\frac{\omega_{it}}{\gamma_i}$ and $\phi_t = \frac{1}{N}\sum_{i=1}^{N}\frac{\omega_{it}}{\gamma_i^2}$. N corresponds to the number of types.

Consider first the two key pricing variables ξ_t and ϕ_t, as shown in Figure 3. You can immediately notice that the level, slope, and curvature of both of these measures changes substantially for different numbers of preference types. Additionally you can notice that the heterogeneity wedge, defined as $\frac{\phi(t)}{\xi(t)^2}$ is also changing. Changing the mass of agents over the support changes the rate of convergence, as in order for the least risk averse agent to dominate they must accumulate a greater consumption share to bring ξ_t and ϕ_t to the same long run level.

In Figure 4 you can see that the level, slope, and curvature of the interest rate and market price of risk are different for different numbers of preference types. For two types the convergence is fast. Both variables rise and their convergence becomes slower as we increase the number of agents, implying a more long-lived volatility in asset pricing

[3]For details on the solution method, see appendix B.

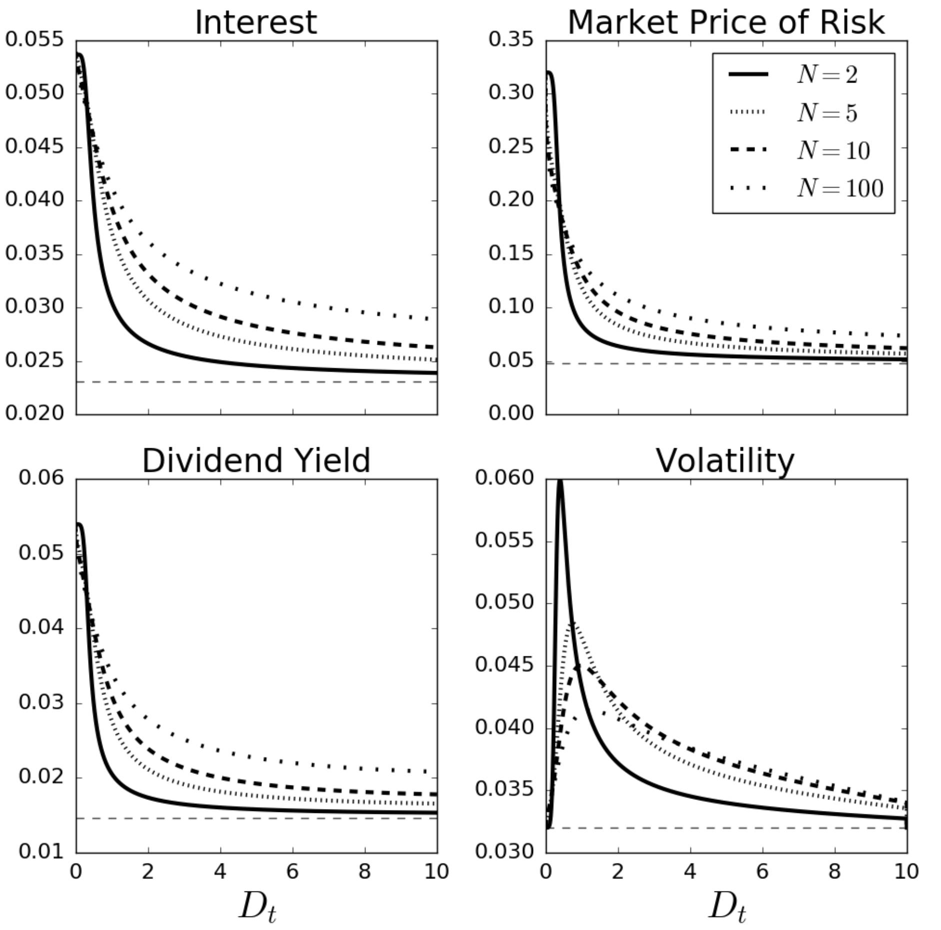


Fig. 4. Interest rate, market price of risk, dividend yield, and volatility for different numbers of agents.

variables. This volatility is consistent with what we observe in financial markets, in that the interest rate and market price of risk do indeed vary with aggregate output. In addition, the change in levels is substantial, with the interest rate being 100 basis points higher at $D = 2.0$ for 100 agents than for 2 agents.

Changes in the interest rate and market price of risk caused by changes in the number of types have noticeable effects on risky asset prices. Asset prices fall in the face of higher levels of the interest rate and market price of risk, while volatility rises (Figure 4). More preference types drives down the average EIS. Meanwhile, more types reduces the speed of convergence, causing small shocks to produce larger changes in asset prices and increasing the volatility. In addition, volatility has a large spike for low values of the dividend when there is less heterogeneity, implying that downside volatility is greater when preference heterogeneity is low. Beyond simply the level, it is interesting to note the slope and curvature of volatility. A negative shock increases volatility, while a positive shock reduces volatility to a lesser degree. This points to one possible explanation of the short-coming often noted in the celebrated work of Black and Scholes (1973), namely constant volatility. The implied volatility estimated from options pricing data exhibits a similar negative slope and convex shape, known as the "volatility smile"[4]. This model produces such stochastic volatility and, in particular, greater downside volatility. In addition to stochastic volatility, the dividend yield co-moves with aggregate consumption, implying predictability in stock market returns as observed in the data by Campbell and Shiller (1988a,b). This is driven by co-movement between the SDF and aggregate consumption, similar to that observed in Mankiw (1981).

On the bond side, we see a non-zero and stochastic supply of credit, creating "leverage cycles". Increasing the number of agents increases leverage, as shown in Figure 5. This is driven by two forces. More preference types means a larger supply of bonds. At the same time, more types reduces asset prices, reducing the value of agents' collateral and further increasing leverage. However, notice that a rise in D_t generates a fall in leverage and vice-versa. This implies a counter-cyclical leverage cycle, opposite that postulated in the literature (e.g. Fostel and Geanakoplos (2008)). This contradiction arises from a difference in the response of borrowing and asset prices. Both are positively correlated with the dividend, but since asset prices are more volatile a fall in the dividend causes a greater fall in asset prices, which increases total leverage.

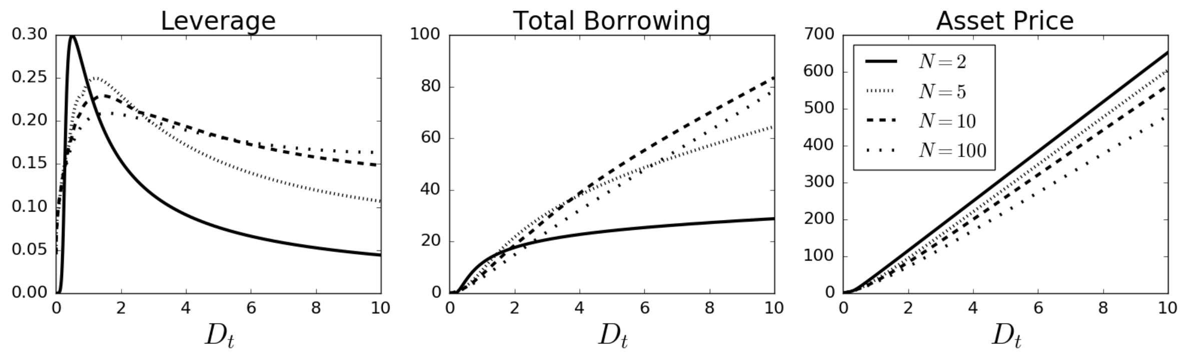


Fig. 5. Leverage, total borrowing, and asset prices for different numbers of agents.

In summation, heterogeneous preferences can explain several qualitative features about financial markets, but the choice of the number of preference types is important if one believes there is a continuum of types.

4. Conclusion

In this paper I have studied how the distribution of risk preferences affects financial variables, consumption shares, and portfolio decisions. The distribution of risk preferences has a large effect on financial variables driven mainly by consumption weighted averages of the EIS. In addition increasing the number of preference types can provide an explanation for the equity risk premium and risk free rate puzzles of Mehra and Prescott

[4]See Lorig and Sircar (2016) for a nice review or Guyon and Henry-Labordère (2013) for a thorough mathematical treatment of several modeling approaches.

(1985); Weil (1989). This is caused by hetereogeneity allowing markets to clear at different marginal preference levels. Agents' relative position to these marginal preference levels determines four groups of market participants: investors, divestors, borrowers, and lenders.

The model can produce both pro- and counter-cyclical leverage cycles depending on the distribution of preferences. The cyclicality of the leverage cycle is driven by the volatility of total borrowing and asset prices. When total borrowing is more volatile than asset prices, leverage cycles are pro-cyclical, and vice-versa. In order for total borrowing to be volatile, there needs to be a large mass of lenders and a small, wealth-poor, group of low risk aversion borrowers.

Additionally, dividend yield in this model co-moves negatively with the growth rate in dividends. This co-movement implies a predictable component in stock market returns. Dividends fall when a negative shock hits the economy and the distribution of consumption shares shifts towards more risk averse agents. This shift reduces asset prices and predicts a faster growth rate in the future. Papers such as Campbell and Shiller (1988a), Campbell and Shiller (1988b), Mankiw (1981), and Hall (1979) drew differing conclusions about the standard model of asset prices, but, broadly speaking, they all deduced that there was some portion of asset prices that was slightly predictable as a function of the growth rate in aggregate consumption. In the model presented here, we can take a step towards explaining this predictability as the dividend yield co-moves with the SDF.

An interesting direction for future research would be to carry this approach over to incomplete markets, as in Chabakauri (2015), to study how borrowing constraints would affect the accumulation of assets and market dynamics.

References

Achdou, Y., Han, J., Lasry, J.-M., Lions, P.-L., and Moll, B. (2014). Heterogeneous agent models in continuous time. *Preprint*.

Ahn, S., Kaplan, G., Moll, B., and Winberry, T. (2016). No more excuses! efficient computation of heterogeneous agent economies with aggregate shocks.

Arrow, K. J. and Debreu, G. (1954). Existence of an equilibrium for a competitive economy. *Econometrica: Journal of the Econometric Society*, pages 265–290.

Aumann, R. J. (1964). Markets with a continuum of traders. *Econometrica: Journal of the Econometric Society*, pages 39–50.

Barseghyan, L., Molinari, F., ODonoghue, T., and Teitelbaum, J. C. (2015). Estimating risk preferences in the field. Technical report, Mimeo, Department of Economics, Cornell University.

Basak, S. and Cuoco, D. (1998). An equilibrium model with restricted stock market participation. *Review of Financial Studies*, 11(2):309–341.

Bhamra, H. S. and Uppal, R. (2014). Asset prices with heterogeneity in preferences and beliefs. *Review of Financial Studies*, 27(2):519–580.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of political economy*, 81(3):637–654.

Brunnermeier, M. K. and Nagel, S. (2008). Do wealth fluctuations generate time-varying risk aversion? micro-evidence on individuals' asset allocation. *The American Economic Review*, 98(3):713–736.

Campbell, J. Y. and Shiller, R. J. (1988a). The dividend-price ratio and expectations of future dividends and discount factors. *Review of financial studies*, 1(3):195–228.

Campbell, J. Y. and Shiller, R. J. (1988b). Stock prices, earnings, and expected dividends. *The Journal of Finance*, 43(3):661–676.

Cardaliaguet, P., Delarue, F., Lasry, J.-M., and Lions, P.-L. (2015). The master equation and the convergence problem in mean field games. *arXiv preprint arXiv:1509.02505*.

Carmona, R. and Delarue, F. (2013). Probabilistic analysis of mean-field games. *SIAM Journal on Control and Optimization*, 51(4):2705–2734.

Carmona, R., Delarue, F., and Lacker, D. (2014). Mean field games with common noise. *arXiv preprint arXiv:1407.6181*.

Chabakauri, G. (2013). Dynamic equilibrium with two stocks, heterogeneous investors, and portfolio constraints. *Review of Financial Studies*, 26(12):3104–3141.

Chabakauri, G. (2015). Asset pricing with heterogeneous preferences, beliefs, and portfolio constraints. *Journal of Monetary Economics*, 75:21–34.

Chassagneux, J.-F., Crisan, D., and Delarue, F. (2014). A probabilistic approach to classical solutions of the master equation for large population equilibria. *arXiv preprint arXiv:1411.3009*.

Chiappori, P.-A., Gandhi, A., Salanié, B., and Salanié, F. (2012). From aggregate betting data to individual risk preferences.

Coen-Pirani, D. (2004). Effects of differences in risk aversion on the distribution of wealth. *Macroeconomic Dynamics*, 8(05):617–632.

Cozzi, M. (2011). Risk aversion heterogeneity, risky jobs and wealth inequality. Technical report, Queen's Economics Department Working Paper.

Cuoco, D. and He, H. (1994). Dynamic equilibrium in infinite-dimensional economies with incomplete information. Technical report, Working paper, Wharton School, University of Pennsylvania.

Cvitanić, J., Jouini, E., Malamud, S., and Napp, C. (2011). Financial markets equilibrium with heterogeneous agents. *Review of Finance*, page rfr018.

Dumas, B. (1989). Two-person dynamic equilibrium in the capital market. *Review of Financial Studies*, 2(2):157–188.

Epstein, L. G., Farhi, E., and Strzalecki, T. (2014). How much would you pay to resolve long-run risk? *The American Economic Review*, 104(9):2680–2697.

Fostel, A. and Geanakoplos, J. (2008). Leverage cycles and the anxious economy. *The American Economic Review*, 98(4):1211–1244.

Fouque, J.-P., Papanicolaou, G., Sircar, R., and Sølna, K. (2011). *Multiscale stochastic volatility for equity, interest rate, and credit derivatives*. Cambridge University Press.

Gârleanu, N. and Panageas, S. (2015). Young, old, conservative and bold: The implications of heterogeneity and finite lives for asset pricing. *Journal of Political Economy*, 123(3):670–685.

Guvenen, F. (2006). Reconciling conflicting evidence on the elasticity of intertemporal substitution: A macroeconomic perspective. *Journal of Monetary Economics*, 53(7):1451–1472.

Guyon, J. and Henry-Labordère, P. (2013). *Nonlinear option pricing*. CRC Press.

Hall, R. E. (1979). Stochastic implications of the life cycle-permanent income hypothesis: theory and evidence. *NBER Working Paper*, (R0015).

Harrison, J. M. and Pliska, S. R. (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic processes and their applications*, 11(3):215–260.

Kaplan, G., Moll, B., and Violante, G. L. (2016). Monetary policy according to hank. Technical report, Working Paper.

Karatzas, I., Lehoczky, J. P., and Shreve, S. E. (1987). Optimal portfolio and consumption decisions for a small investor on a finite horizon. *SIAM journal on control and optimization*, 25(6):1557–1586.

Karatzas, I. and Shreve, S. E. (1998). *Methods of mathematical finance*, volume 39. Springer Science & Business Media.

Krusell, P. and Smith, Jr, A. A. (1998). Income and wealth heterogeneity in the macroeconomy. *Journal of Political Economy*, 106(5):867–896.

Lasry, J.-M. and Lions, P.-L. (2006a). Jeux à champ moyen. i–le cas stationnaire. *Comptes Rendus Mathématique*, 343(9):619–625.

Lasry, J.-M. and Lions, P.-L. (2006b). Jeux à champ moyen. ii–horizon fini et contrôle optimal. *Comptes Rendus Mathématique*, 343(10):679–684.

Lasry, J.-M. and Lions, P.-L. (2007). Mean field games. *Japanese Journal of Mathematics*, 2(1):229–260.

Lorig, M. and Sircar, R. (2016). Stochastic volatility: Modeling and asymptotic approaches to option pricing & portfolio selection. *Fundamentals of Financial Signal Processing, Wiley (to appear).*

Lucas, R. E. (1978). Asset prices in an exchange economy. *Econometrica: Journal of the Econometric Society*, pages 1429–1445.

Mankiw, N. G. (1981). The permanent income hypothesis and the real interest rate. *Economics Letters*, 7(4):307–311.

Mehra, R. and Prescott, E. C. (1985). The equity premium: A puzzle. *Journal of monetary Economics*, 15(2):145–161.

Merton, R. C. (1969). Lifetime portfolio selection under uncertainty: The continuous-time case. *The review of Economics and Statistics*, pages 247–257.

Merton, R. C. (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of economic theory*, 3(4):373–413.

Moll, B. (2014). Productivity losses from financial frictions: Can self-financing undo capital misallocation? *The American Economic Review*, 104(10):3186–3221.

Press, W. H. (2007). *Numerical recipes 3rd edition: The art of scientific computing.* Cambridge university press.

Weil, P. (1989). The equity premium puzzle and the risk-free rate puzzle. *Journal of Monetary Economics*, 24(3):401–421.

Appendix A. Proofs

Proof of Lemma 1. To solve for the consumption weight of an individual i, take the market clearing condition in consumption and divide through by agent i's consumption, then substitute for individual consumption using Eq. (6)

$$\frac{1}{N}\sum_{j=1}^{N} c_{jt} = D_t \Leftrightarrow c_{it} = \frac{c_{it}}{\frac{1}{N}\sum_{j=1}^{N} c_{jt}} D_t = \left(\frac{N\left(e^{\rho t}\Lambda_i H_t\right)^{\frac{-1}{\gamma_i}}}{\sum_{j=1}^{N}\left(e^{\rho t}\Lambda_j H_t\right)^{\frac{-1}{\gamma_j}}} \right) D_t = \omega_{it} D_t$$

□

Proof of Lemma 3. Model consumption as a geometric Brownian motion:

$$\frac{dc_{it}}{c_{it}} = \mu_{it}^{c} dt + \sigma_{it}^{c} dW_t \tag{22}$$

Solve for H_t in Eq. (6), apply Itô's lemma, and match coefficients to those in Eq. (5), which gives the result. □

Proof of Proposition 1. Recall the definition of consumption dynamics in (22) and the market clearing condition for consumption in (4). Apply Itô's lemma to the market clearing condition:

$$\frac{1}{N}\sum_{i} c_{it} = D_t \Rightarrow \frac{1}{N}\sum_{i} dc_{it} = dD_t$$

By matching coefficients we find

$$\mu_D = \frac{1}{N}\sum_{i} \omega_{it}\mu_{it}^{c} \quad , \quad \sigma_D = \frac{1}{N}\sum_{i} \omega_{it}\sigma_{it}^{c}$$

Now use Lemma 3 to substitute the values for consumption drift and diffusion, then solve for the interest rate and the market price of risk, which gives the result. □

Proof of Proposition 2. Consider the interest rate and market price of risk which prevail in two representative agent economies populated by agents with CRRA parameters γ_{rt} and $\gamma_{\theta t}$, respectively:

$$\begin{aligned} r_t &= \rho + \mu_D \gamma_{rt} - \gamma_{rt}(1+\gamma_{rt})\frac{\sigma_D^2}{2} \\ \theta_t &= \sigma_D \gamma_{\theta t} \end{aligned}$$

Equate these to Eqs. (13) and (14) and solve for γ_{rt} and $\gamma_{\theta t}$. To find the inequality, take

$\gamma_{rt} \lesseqgtr \gamma_{\theta t}$ which is equivalent to

$$\pm\sqrt{\frac{\mu_D}{\sigma_D^2}\left(\frac{\mu_D}{\sigma_D^2} - 1 - \frac{2}{\xi_t}\right) + \frac{\xi_t + \phi_t}{\xi_t^3} + \frac{1}{4}} \lesseqgtr \frac{1}{\xi_t} + \frac{1}{2} - \frac{\mu_D}{\sigma_D^2}$$

If we take only the negative root on the left hand side, we find no matter the sign on the right that $\gamma_{rt} \leq \gamma_{\theta t}$. In the case of the positive root $\gamma_{rt} \geq \gamma_{\theta t}$, but this marginal preference level tends to be very high. There could exist settings where this positive root exists in the support, but for simplicity I consider in this paper only settings where this root lies outside of the support. □

Proof of Proposition 3. Assume that consumption weights follow a geometric Brownian motion given by

$$\frac{d\omega_{it}}{\omega_{it}} = \mu_{it}^{\omega} dt + \sigma_{it}^{\omega} dW_t \tag{23}$$

Apply of Itô's lemma to the definition of consumption weights in (**??**):

$$\omega_{it} = \frac{\left(\Lambda^i e^{\rho t} H_t\right)^{\frac{-1}{\gamma_i}}}{\sum_j \left(\Lambda_j e^{\rho t} H_t\right)^{\frac{-1}{\gamma_j}}} \Leftrightarrow \omega^i(t) = \left[\sum_j \Lambda_j^{\frac{-1}{\gamma_j}} \Lambda_i^{\frac{1}{\gamma_i}} \left(e^{\rho t} H_t\right)^{\frac{1}{\gamma_i} - \frac{1}{\gamma_j}}\right]^{-1}$$

Matching coefficients gives the result. □

Proof of **??**. Assume there exists a Markovian equilibrium in D_t. Then an individual's Hamilton-Jacobi-Bellman (HJB) equation writes

$$\begin{aligned} 0 = \max_{c_{it}, \pi_{it}} \Bigg\{ & e^{-\rho t} \frac{c_{it}^{1-\gamma_i} - 1}{1 - \gamma_i} + \frac{\partial J_{it}}{\partial t} + \left[X_{it}\left(r_t + \pi_{it}\sigma_t\theta_t\right) - c_{it}\right] \frac{\partial J_{it}}{\partial X_{it}} \\ & + \mu_D D_t \frac{\partial J_{it}}{\partial D_t} + \sigma_D \sigma_t \pi_{it} D_t X_{it} \frac{\partial^2 J_{it}}{\partial X_{it} \partial D_t} + \frac{1}{2}\left[X_{it}^2 \pi_{.it}^2 \sigma_t^2 \frac{\partial^2 J_{it}}{\partial X_{it}^2} + \sigma_D^2 D_t^2 \frac{\partial^2 J_{it}}{\partial D_t^2}\right] \Bigg\} \end{aligned} \tag{24}$$

subject to the transversality condition $\lim_{t\to\infty} \mathbb{E}_t J_{it} = 0$ for all i s.t. $\gamma_i > \underline{\gamma}$, as the agent with the lowest risk aversion will dominate in the long run (Cvitanić et al. (2011)). First order conditions imply

$$c_{it} = \left(e^{-\rho t} \frac{\partial J_{it}}{\partial X_{it}}\right)^{\gamma_i} \tag{25}$$

$$\pi_{it} = -\left(X_{it}\sigma_t \frac{\partial^2 J_{it}}{\partial X_{it}^2}\right)^{-1} \left[\theta_t \frac{\partial J_{it}}{\partial X_{it}} + \sigma_D D_t \frac{\partial^2 J_{it}}{\partial X_{it} \partial D_t}\right] \tag{26}$$

Assume that the value function is separable as

$$J_{it}(X_{it}, D_t) = e^{-\rho t} \frac{X_{it}^{1-\gamma_i} V_i(D_t)^{\gamma_i}}{1 - \gamma_i} \tag{27}$$

Substituting Eq. (27) into Eqs. (25) and (26) gives

$$c_{it} = \frac{X_{it}}{V_i(D_t)} \tag{28}$$

$$\pi_{it} = \frac{1}{\gamma_i \sigma_t}\left(\gamma_i \sigma_D D_t \frac{V_i'(D_t)}{V_i(D_t)} + \theta_t\right) \tag{29}$$

which shows that $V_i(D_t)$ is the wealth-consumption ratio as a function of the dividend. Next, substitute Eqs. (27) to (29) into Eq. (24) and simplify to find

$$\begin{aligned} 0 =& 1 + \frac{\sigma_D^2 D_t^2}{2} V_i''(D_t) + \left[\frac{1-\gamma_i}{\gamma_i}\theta_t \sigma_D + \mu_D\right] D_t V_i'(D_t) \\ &+ \left[(1-\gamma_i)r_t - \rho + \frac{1-\gamma_i}{2\gamma_i}\theta_t^2\right]\frac{V_i(D_t)}{\gamma_i} \end{aligned} \tag{30}$$

Which gives an ode for the wealth-consumption ratio. On the boundaries $D = 0$ and $D \to \infty$, the most risk averse and the least risk averse agent dominates, respectively (Cvitanić et al. (2011)). The boundary conditions correspond to the value functions for individuals when prices are set by these dominant agents. One could identically take the limit in the ODE itself to arrive at the boundary conditions.

Define the price-dividend ratio as a function of the single state variable: $\mathcal{S}(D_t) = \frac{S_t}{D_t}$. Apply Itô's lemma to $D_t\mathcal{S} = S_t$ and match coefficients to find

$$\begin{aligned} \mu_t &= D_t^2 + \frac{(\sigma_D D_t)^2}{2}\frac{\mathcal{S}''(D_t)}{\mathcal{S}(D_t)} D_t + D_t \mu_D + \frac{\mathcal{S}'(D_t)}{\mathcal{S}(D_t)}(\sigma_D D_t)^2 \\ \sigma_t &= \sigma_D\left(1 + D_t \frac{\mathcal{S}'(D_t)}{\mathcal{S}(D_t)}\right) \end{aligned}$$

Taking the market clearing condition for wealth, rewrite $\mathcal{S}(D_t)$ as a function of D_t:

$$S_t = \frac{1}{N}\sum_i X_{it} \Leftrightarrow \frac{S_t}{D_t} = \mathcal{S}(D_t) = \frac{1}{N}\sum_i \frac{X_{it}}{D_t} = \frac{1}{N}\sum_i \frac{X_{it}}{c_{it}}\frac{c_{it}}{D_t} = \frac{1}{N}\sum_i V(D_t)\omega_{it}$$

which gives $\mathcal{S}(D_t)$ given that $\omega_{it} = f_i(D_t)$

□

Proof of Proposition 6. This proof proceeds identically to Chabakauri (2015). Let $V_i(D) \in C^1[0,\infty) \cup C^2[0,\infty)$, $0 < V_i(D) \le C_1$, $|\pi_{it}\sigma_t| < C1$, and $|\theta_t| < C_1$, where C_1 is a constant.

Additionally, assume

$$\mathbb{E}\int_0^\infty e^{-\rho t}\frac{c_{it}^{1-\gamma_i}}{1-\gamma_i}dt < \infty \tag{31}$$

$$\mathbb{E}\int_0^T J_i(X_{it}, D_t, t)^2 dt < \infty \qquad \forall\ T > 0 \tag{32}$$

$$\limsup_{T\to\infty} \mathbb{E}J_i(X_{it}, D_t, t) \geq 0 \tag{33}$$

Define $U_t = \int_0^t e^{-\rho t} c_{i\tau}^{1-\gamma_i}/(1-\gamma_i)d\tau + J_i(X_{it}, D_t, t)$, which satisfies $dU_t = \mu_{Ut}dt + \sigma_{Ut}dW_t$ such that

$$\mu_{Ut} = \left(e^{-\rho t}\frac{c_{it}^{1-\gamma_i}-1}{1-\gamma_i} + \frac{\partial J_{it}}{\partial t} + \left[X_{it}\left(r_t + \pi_{it}\sigma_t\theta_t\right) - c_{it}\right]\frac{\partial J_{it}}{\partial X_{it}} + \mu_D D_t \frac{\partial J_{it}}{\partial D_t}\right. \tag{34}$$

$$\left. + \sigma_D\sigma_t\pi_{it}D_tX_{it}\frac{\partial^2 J_{it}}{\partial X_{it}\partial D_t} + \frac{1}{2}\left[X_{it}^2\pi_{.it}^2\sigma_t^2\frac{\partial^2 J_{it}}{\partial X_{it}^2} + \sigma_D^2 D_t^2\frac{\partial^2 J_{it}}{\partial D_t^2}\right]\right) \tag{35}$$

$$\sigma_{Ut} = J_{it}\left((1-\gamma_i)\pi_{it}\sigma_t + \gamma_i D_t\sigma_D\frac{V_i'(D)}{V_i(D)}\right) = J_{it}\left(\pi_{it}\sigma_t - \theta_t\right) \tag{36}$$

Notice that μ_{Ut} is simply the PDE inside the max operator in the HJB Eq. (24), thus $\mu_{Ut} \leq 0$. By the boundedness conditions, U_t is integrable and because its drift is negative it is a supermartingale, thus $U_t \geq \mathbb{E}_t U_T\ \forall\ t \leq T$, which is equivalent to

$$J_i(X_{it}, D_t, t) \geq \mathbb{E}_t\int_t^T e^{-\rho(\tau-t)}\frac{c_{i\tau}^{1-\gamma_t}}{1-\gamma_i}d\tau + \mathbb{E}_t J_i(X_{it}, D_t, T) \tag{37}$$

Since the first term is monotonic in T, by Eq. (33) and by the monotone convergence theorem we have

$$J_i(X_{it}, D_t, t) \geq \mathbb{E}_t\int_t^\infty e^{-\rho(\tau-t)}\frac{c_{i\tau}^{1-\gamma_t}}{1-\gamma_i}d\tau \tag{38}$$

Now to show the opposite, consider first the limit $\mathbb{E}_t J_i(X_{i\tau}, D_\tau, \tau) \to 0$ as $\tau \to \infty$. Applying Itô's lemma to $J_i(X_{it}, D_t, t)$ and following similar steps as before, we find $dJ_{it} = J_{it}[\mu_{Jt}dt + \sigma_{Jt}dW_t]$ where

$$\mu_{Jt} = \frac{-1}{V_i(D)}$$

$$\sigma_{Jt} = \pi_{it}\sigma_t - \theta_t$$

by the first order conditions Eqs. (28) and (29). By the boundedness assumptions σ_{Jt} satisfies Novikov's conditions and we have that $d\eta_t = \eta_t\sigma_{Jt}dW_t$ acts as a change of measure to remove the Brownian term in J_{it}. We have

$$|\mathbb{E}_t J_i(X^*_{i\tau}, D_\tau, \tau)| \leq \mathbb{E}_t \left[|J_{i\tau}| exp\left\{ -\int_t^T \frac{1}{V_i(D_u)} du \right\} \frac{\eta_\tau}{\eta_t} \right]$$
$$\leq |J_{it}| e^{-(T-t)/C_1} \mathbb{E}_t \frac{\eta_\tau}{\eta_t} = |J_{it}| e^{-(T-t)/C_1}$$

Taking the limit in T gives the result.

Finally, define U^*_t as for U_t, except evaluated at the optimum consumption. Then

$$dU^*_t = J_{it} \left(\pi_{it} \sigma_t - \theta_t \right) dW_t \tag{39}$$

Again applying Novikov's condition we get that U^*_t is an exponential martingale, which gives

$$J_i(X_{it}, D_t, t) = \mathbb{E}_t \int_t^T e^{-\rho(\tau - t)} \frac{(c^*_{it})^{1-\gamma_t}}{1-\gamma_i} d\tau + \mathbb{E}_t J_i(X^*_{it}, D_t, T)$$

Finally, by the intermediate result the last term goes to zero, showing that we do indeed have the optimum. □

Appendix B. Numerical Methods

B.1. ODE Solution by Finite Difference

To solve for portfolios and wealth, one needs to solve the ode in Eq. (17). In this work I use finite difference methos (see Press (2007)). In the following I suppress the i subscript for clarity. Using a central difference scheme (assuming an evenly spaced grid), the ode for a given i can be approximated as

$$0 = 1 + a(D_k) \frac{V_{k+1} - 2V_k + V_{k-1}}{h^2} + b(D_k) \frac{V_{k+1} - V_{k-1}}{2h} + c(D_k) V_k$$

where D_k corresponds to the kth point in the grid, h the step size, $a(D_k) = \sigma_D^2 D_k^2/2$, $b(D_k) = ((1-\gamma_i)\theta(D_k)\sigma_D/\gamma_i + \mu_D)D_k$, $c(D_k) = ((1-\gamma_i)r(D_k) - \rho + (1-\gamma_i)\theta(D_k)^2/(2\gamma_i))/\gamma_i$. This can be rewritten as a system of linear equations:

$$0 = 1 + (x_k - y_k)V_{k-1} + (z_k - 2x_k)V_k + (x_k + y_k)V_{k+1}$$

where $x_k = a(D_k)/h^2$, $y_k = b(D_k)/2h$, and $z_k = c(D_k)$. Combining this system of equations with the boundary conditions in Eq. (18) one gets a system of $K - 2$ equations in $K - 2$ unknowns which takes a highly spars structure (Note: This paper takes the approach of fixing $lim_{D\to\infty} V'(D) = 0$, or a reflecting boundary condition. This seems to provide more stability and is confirmed by numerical simulations, although may not be the "best" approximation). This paper uses `scipy.sparse` to build the matrix equation and solve for the value functions.

Appendix C. Extension to Infinite Types

Consider the limiting case as $N \to \infty$. If we take the consumption weights, $\omega_{it} = \omega_t(\gamma_i, \alpha_i)$, we have a function of an empirical mean, which converges to the mean with respect to the initial distribution by the Strong Law of Large Numbers:

$$\omega_t(\gamma_i, \alpha_i) = \frac{N\left(\Lambda(\gamma_i, \alpha_i)e^{\rho t}H_t\right)^{\frac{-1}{\gamma_i}}}{\sum_{j=1}^{N}\left(\Lambda(\gamma_j, \alpha_j)e^{\rho t}H_t\right)^{\frac{-1}{\gamma_j}}} \underset{N\to\infty}{\to} \frac{\left(\Lambda(\gamma, \alpha)e^{\rho t}H_t\right)^{\frac{-1}{\gamma}}}{\int\left(\Lambda(\gamma, \alpha)e^{\rho t}H_t\right)^{\frac{-1}{\gamma}} dG(\gamma, \alpha)} = \omega_t(\gamma, \alpha)$$

This result is logical when viewed through the lens of work on a continuum of agents à la Aumann (1964). However, the market clearing condition for consumption weights implies something intriguing about their relationship to the initial distribution. If we think of consumption weights as a ratio of probability measures, then they act as the Radon-Nikodym derivative of a stochastic measure with respect to the distribution of the initial condition. That is, define $\omega_t(\gamma, \alpha) = \frac{dP_t(\gamma,\alpha)}{dG(\gamma,\alpha)}$. Then we have

$$\int \omega_t(\gamma, \alpha)dG(\gamma, \alpha) = \int \frac{dP_t(\gamma, \alpha)}{dG(\gamma, \alpha)}dG(\gamma, \alpha) = \int dP_t(\gamma, \alpha) = 1$$

The evolution of this distribution would be difficult to describe directly, but the expressions in Proposition 3 give the dynamics of this stochastic distribution. So $\omega_t(\gamma, \alpha)$ allows one to calculate exactly the evolution of this stochastic distribution by use of a change of measure. Alternatively, one can think of $\omega_t(\gamma, \alpha)$ as a sort of importance weight, where as the share of risky assets is concentrated towards one area in the support, the weight of this area grows in the determination of asset prices.

Additionally, the Radon-Nikodym interpretation allows one the accuracy of finite types as an approximation to continuous types. Say for instance we would like to discretize the above expression for the market clearing condition on $\omega_t(\gamma, \alpha)$ using a Riemann sum with an evenly spaced partition (e.g. a midpoint rule):

$$\int \omega_t(\gamma, \alpha)dG(\gamma, \alpha) \approx \frac{(\overline{\gamma} - \underline{\gamma})(\overline{\alpha} - \underline{\alpha})}{JK} \sum_{k=1}^{K}\sum_{j=1}^{J} \omega_t(\gamma_k, \alpha_j)g(\gamma_k, \alpha_j) \tag{40}$$

This looks quite similar to the market clearing conditions in the finite type model (Eq. (4)). Make the identification $N = JK$ and notice that since $\omega_t(\gamma, \alpha)$ is a geometric Brownian motion, $\omega_t(\gamma, \alpha) = \omega_0(\gamma, \alpha)\hat{\omega}_t(\gamma, \alpha)$ where $\hat{\omega}_t(\gamma, \alpha)$ is a stochastic process with initial value 1. If we define the initial condition on omega as $\omega_0(\gamma,\alpha) = \frac{1}{(\overline{\gamma}-\underline{\gamma})(\overline{\alpha}-\underline{\alpha})g(\gamma,\alpha)}$, then Eq. (40) becomes

$$\int \omega_t(\gamma,\alpha)dG(\gamma,\alpha) \approx \frac{1}{N}\sum_{k=1}^{K}\sum_{j=1}^{J}\hat{\omega}_t(\gamma_k,\alpha_j) \tag{41}$$

Now Eq. (41) matches exactly the condition in Eq. (4). However, this equation has particular implications about the Radon-Nikodym derivative. From the definition of the Radon-Nikodym derivative we can write

$$P_t(A) = \int_A \omega_t(\gamma,\alpha)dG(\gamma,\alpha) = \int_A \omega_t(\gamma,\alpha)g(\gamma,\alpha)d\gamma d\alpha$$

Substituting the imposed definition of $\omega_t(\gamma,\alpha)$ we have

$$P_t(A) = \int_A \frac{\hat{\omega}_t(\gamma,\alpha)}{(\overline{\gamma}-\underline{\gamma})(\overline{\alpha}-\underline{\alpha})}d\gamma d\alpha$$

Now, since $\hat{\omega}_0(\gamma,\alpha) = 1$, the above implies

$$P_0(A) = \int_A \frac{1}{(\overline{\gamma}-\underline{\gamma})(\overline{\alpha}-\underline{\alpha})}d\gamma dx = \int_A \omega_0(\gamma,\alpha)g(\gamma,\alpha)d\gamma d\alpha$$

This implies that the evenly spaced grid approximation of preference distributions produces a particular assumption about the initial condition $\omega_0(\gamma,\alpha)g(\gamma,\alpha)$. The initial condition in such an approximation is limited to a uniform distribution such that $\omega_0(\gamma,\alpha)g(\gamma,\alpha) = \frac{1}{(\overline{\gamma}-\underline{\gamma})(\overline{\alpha}-\underline{\alpha})}$. Any change in initial consumption weights will change the underlying assumptions about the distribution $g(\gamma,\alpha)$.

The continuous types model provides several useful modeling tools beyond finite types. First, the joint distribution of initial wealth and risk aversion is explicitly modeled. In a model of finite types over an evenly spaced grid one can only model a product distribution such that $\omega_0(\gamma,\alpha)g(\gamma,\alpha)$ is uniform. Second, but closely related, is the computational simplification provided by the continuum. One can simulate quadrature points to approximate a continuous distribution, whereas to do the same for the finite types model would require many simulated agents. Finally, the continuum allows one to coherently model the distribution of risk preferences if one believes there to be many preference types and if the number of types has an effect on model predictions.

Appendix D. Finite Types versus Continuous Types Simulation

For the continuous types model we must approximate the integrals in some way. For simplicity I use a trapezoidal rule. As an example, consider the definition of $\xi(t)$ and its associated quadrature approximation when there is only a single initial endowment type:

$$\xi_t = \int \frac{\omega(t,\gamma,\alpha)}{\gamma} dG(\gamma,\alpha) \approx \frac{\overline{\gamma} - \underline{\gamma}}{2(M-1)} \sum_{m=1}^{M-1} \left[\frac{\omega_t(\gamma_m) g(\gamma_m)}{\gamma_m} + \frac{\omega_t(\gamma_{m+1}) g(\gamma_{m+1})}{\gamma_{m+1}} \right]$$

where (γ_m) is an evenly spaced grid. For finite types, changing the number of simulated points changes the distribution $g(\gamma)$ in the model, while for the continuous types solution, changing the number of quadrature points does not change the assumptions about $g(\gamma)$. This will be the key feature that differentiates the two solutions.

Consider the same uniform distribution as before. In this case the results look almost identical to the finite types case. In Figure 6 you can see that the interest rates are very similar. This similarity is driven by the integral approximations and the fact that we are using a uniform distribution over preferences. As pointed out in section C, we could *only* match a uniform initial distribution. To see this we can look at robustness results for a more complex assumption about the distribution of preferences and different methods of approximation.

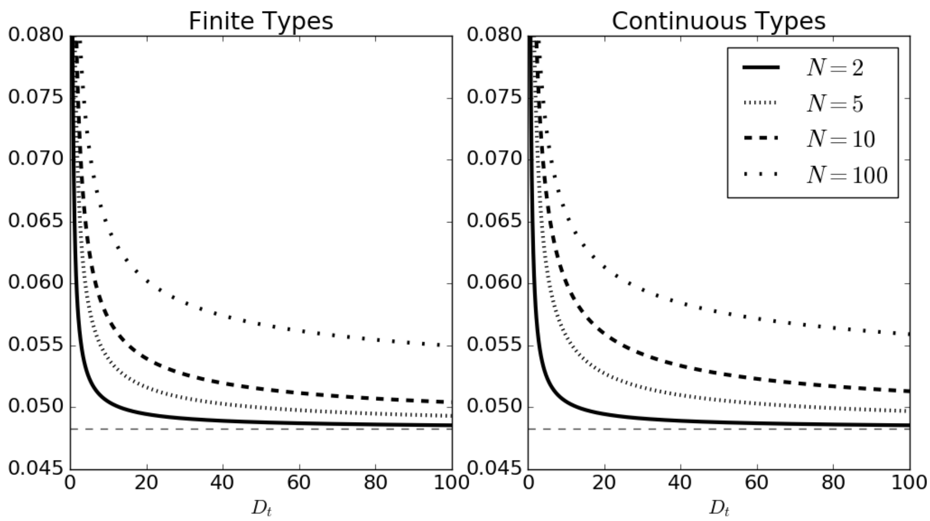


Fig. 6. Interest rates for different numbers of agents and quadrature points, respectively, under the assumption of a uniform distribution of preferences.

Appendix E. Simulation Robustness

Consider the case where preferences follow a Beta(2, 2) distribution over the same support. In this case, we could take several approaches to solving the finite types model. First, we could consider taking the same naive uniform approximation, fixing a uniform distribution of consumption weights over the same support. Second, we could consider initializing the consumption weights to match the same initial condition in the continuous types case, i.e. $\omega_0(\gamma) = g(\gamma)$. Third, we could attempt to use a monte-carlo approximation, drawing many agents from the distribution $g(\gamma)$. Finally, we could use the continuous types model, directly. The interest rate is presented in Figure 7 for each of these approximations.

You will notice that the solutions are substantially different. First, a uniform distribution of agents is a poor approximation to a non-uniform distribution of preferences (as one would expect). Second, the monte-carlo approximation converges to the continuous types solution very slowly, but is much better than the uniform approximation. Finally, one might think that changing the initial condition in $\omega_t(\gamma_i)$ to match the distribution of preferences one could recover the same solution. However, this simultaneously changes the assumptions about the distribution $g(\gamma)$, as was pointed out in section C, producing a noticeably different interest rate.

In addition, the assumption of a Beta distribution over preferences changes the outcome for financial variables. You'll see in Figure 8 that not only is volatility substantially higher for a longer period of time, but leverage is as well. Additionally, both variables exhibit an inflection point. For very low values of D_t leverage cycles become pro-cyclical and the volatility smile inverts. This difference with the uniform distribution case is driven by the volatility in borrowing. In order to produce a pro-cyclical leverage cycle, total borrowing must fall more quickly than total wealth. In this case, the less risk averse agent is constrained by their wealth being very low and thus limiting their borrowing. When dividends become low enough, their demand for borrowing collapses. In addition, pro-cyclical leverage cycles are accompanied by pro-cyclical volatility, or an inverted volatility skew. This comovement represents a testable implication.

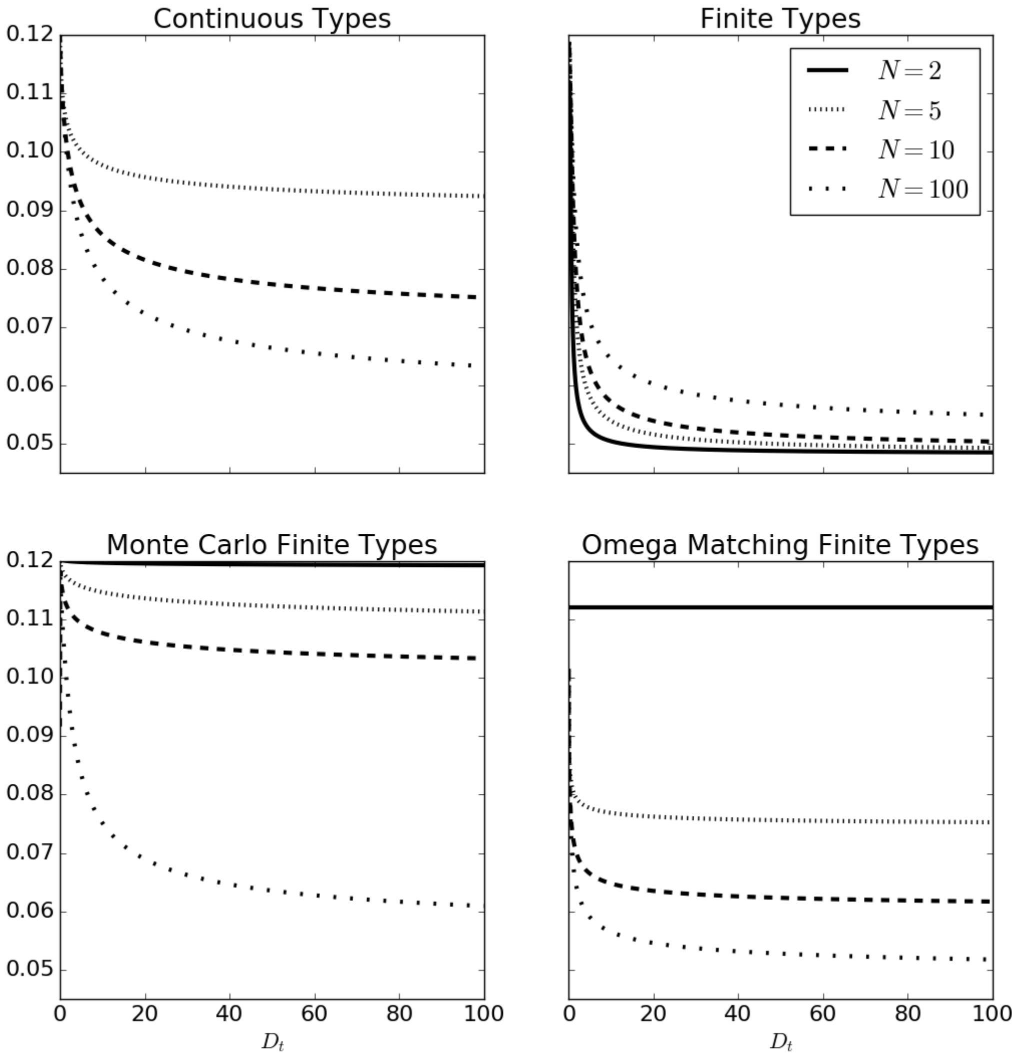


Fig. 7. Interest rate for different numbers of points and different approximation strategies assuming preferences distributed Beta(2, 2).

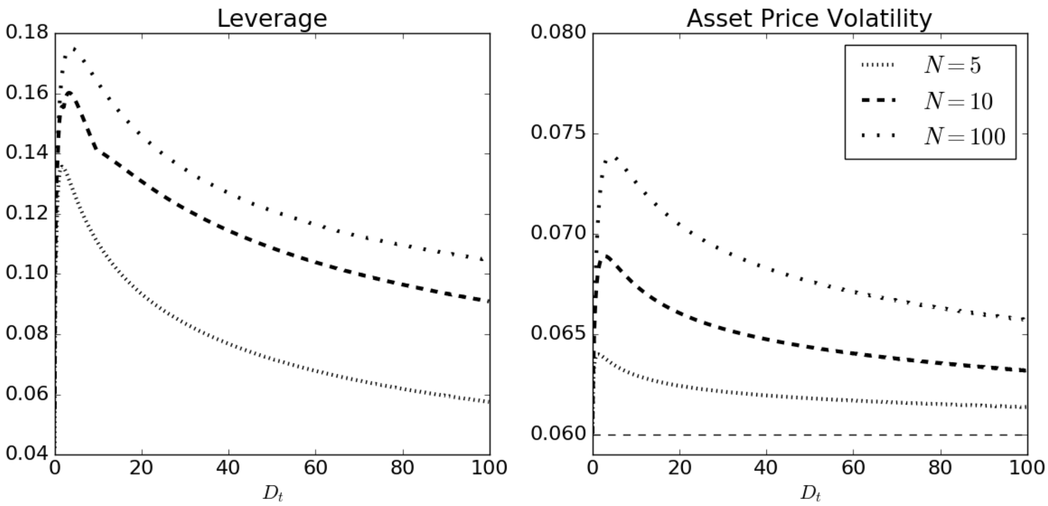


Fig. 8. Leverage and volatility for different numbers of quadrature points assuming continuous types distributed beta(2, 2).