Heterogeneous Preferences, Constraints, and the Cyclicality of Leverage

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Abstract

This paper documents a new stylized fact about the leverage cycle and proposes a model of risk preference heterogeneity to explain this fact. In particular, leverage cyclicality depends on the preferences of the marginal agent in the economy, as this determines financial variables. I propose a model of risk preference heterogeneity to explain this fact and prove existence of a new, low dimensional Markovian equilibrium which may exist in other heterogeneous agent models. This equilibrium is studied in an application to margin constraints. It is shown how this type of constraint increases the market price of risk and decreases the interest rate, producing a higher equity risk premium and asset price bubbles. In addition, heterogeneity and margin constraints are shown to produce both pro- and counter-cyclical leverage cycles as seen in the data. Finally, more preference types causes a reduction in the severity of crisis and a lower relative deviation from complete markets in all variables. At the same time expected returns on the stock must remain high to compensate risk averse agents to hold a larger share.

Keywords: Asset Pricing, Heterogeneous Agents, General Equilibrium, Financial Economics.

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Introduction

Financial leverage has become an important policy variable since the crisis of 2007-2008. In particular leverage allows investors to increase the volatility of balance sheet equity, producing the possibility of greater returns. At the same time leveraged investors are exposed to larger down-side risk. In the face of negative shocks, constrained investors must sell assets to reduce their leverage. This is known as the "leverage cycle". The associated credit contraction produces great volatility in asset prices and has been the target of regulation in the post-crisis era (e.g. the Basel III capital requirement rules). However, leverage' cyclicality remains a topic of debate. In this paper I document that cycles can be both pro- and counter-cyclical. In addition I characterize the equilibrium for an incomplete market model of heterogeneous risk preferences and prove the existence of a previously unknown, low-dimensional Markovian equilibrium which may be applicable to more general settings. Finally, I simulate the model and find that the severity of crisis is reduced when agents are more diverse.

In theoretical models leverage cyclicality depends greatly on the underlying assumptions producing trade. In his foundational work on the topic, Geanakoplos (1996) shows how the combination of belief heterogeneity and margin constraints produce a pro-cyclical leverage cycle. However, this finding is in opposition to the contemporary paper by Kiyotaki and Moore (1997), where participation constraints force agents to invest through intermediaries, whose credit constraints produce a counter-cyclical leverage cycle. More recently He and Krishnamurthy (2013) and Brunnermeier and Sannikov (2014) also produce counter-cyclical leverage cycles by including intermediaries through whom constrained agents can profit from risky assets. In fact, He and Krishnamurthy (2013) even points out the debate in the applied literature and the fact that, "[Their] model does not capture the other aspects of this process, ... that some parts of the financial sector reduce asset holdings and deleverage."

This ambiguity over the cyclicality of leverage has been noted in different ways by the empirical literature. Korajczyk and Levy (2003) study the capital structure of firms and finds that leverage is counter-cyclical for unconstrained firms and pro-cyclical for constrained firms. However, Halling et al. (2016) contradicts this by showing that target leverage is counter-cyclical onces you account for variation in explanatory variables, pointing out that the effect in Korajczyk and Levy (2003) is only the "direct effect". In the cross section of the economy Adrian and Shin (2010b) find that leverage is counter-cyclical for households, ambiguous for non-financial firms, and pro-cyclical for broker dealers. However, the authors study leverage against changes in balance sheet assets. This comparison produces a mechanical correlation which somewhat disappears when assets are replaced by GDP growth as a proxy for the business cycle (see Figure 2). Ang et al. (2011) point out that when accounting for prices broker dealer leverage is counter-cyclical, but that hedge fund leverage is pro-cyclical. These contrary studies can be reconciled when controlling for financial variables such as the price/dividend ratio or the interest rate. In fact, for several sectors studied (see section 1) the leverage cycle is *both* pro- and counter-cyclical. This ambiguity can be explained by a model of preference heterogeneity.

Many authors have criticized the assumption of a representative, constant relative risk aversion agent since the Mehra and Prescott (1985) posited the equity risk premium puzzle. Other utility functions were the first major response to this puzzle, in particular Epstein-Zin preferences (Epstein and Zin (1989); Weil (1989)) and habit formation (Campbell and Cochrane (1999)) have been used to explain this and other puzzles. However, several papers have studied preferences across individuals and found them to be heterogeneous and constant in time (Brunnermeier and Nagel (2008); Chiappori and Paiella (2011); Chiappori et al. (2012)). In addition, Epstein et al. (2014) pointed out that the assumptions necessary to match the risk premium using Epstein-Zin preferences produce unrealistic preference for early resolution of uncertainty. Beyond these criticisms, one needs heterogeneity in order to generate leverage. Risk preference heterogeneity has succeeded in partially responding to these issues.

Heterogeneity in risk preferences has been used to generate trade in financial models since the foundational paper of Dumas (1989). Since then many authors have studied the problem from different angles, assuming different levels of market completeness, utility functions, participation constraints, information structures, etc., but almost always under the assumption of only two preference types (Basak and Cuoco (1998); Coen-Pirani (2004); Guvenen (2006); Kogan et al. (2007); Guvenen (2009); Cozzi (2011); Garleanu and Pedersen (2011); Rytchkov (2014); Longstaff and Wang (2012); Prieto (2010); Christensen et al. (2012); Bhamra and Uppal (2014); Chabakauri (2013, 2015); Gârleanu and Panageas (2015); Santos and Veronesi (2010)). Cvitanić et al. (2011) studies the problem of N agents with several dimensions of heterogeneity and focuses on the dominant agents, characterizing portfolios via the Malliavan calculus. Abbot (2017) studies a similar setting with N heterogeneous CRRA agents in a complete financial market using a value function approach and shows how changes in the number of types can produce substantially different quantitative results and how the variance in preferences provides an additional degree of freedom for explaining the equity risk premium puzzle. However, that work produces large amounts of leverage and high margins. This observation points towards the need to introduce some degree of constraint or incompleteness to better match the real world. To that end, this paper studies the same type of economy with N heterogeneous CRRA agents under convex portfolio constraints with an application to margin constraints.

Margin constraints and preference heterogeneity generate both pro- and counter-cyclical leverage cycles. When aggregate production is high risk neutral agents dominate the economy and the price of risky assets is high. High asset prices increase individual wealth and reduce leverage. When aggregate production is low risk averse agents dominate, reducing asset prices. Low asset prices cause individual wealth to be low and individual leverage to be high. With a margin constraint risk neutral agents eventually run into a borrowing limit. This limit forces risk averse agents to hold more risky assets and pushes up asset prices. In turn, total leverage falls. In this way heterogeneous preferences and margin constraints produce both pro- and counter-cyclical leverage cycles. This effect is robust to different assumptions about the distribution of preferences, but the size of the cycle depends on the number of preference types.

The degree of heterogeneity is found to reduce the severity of crises for two reasons. First, more preference types implies less severe swings in financial variables even when markets are complete. The marginal agent changes more slowly in the face of aggregate shocks when there is greater heterogeneity. Second, the size of the shock is affected by the mass of agents facing a marginally binding constraint. The shock is large when the mass of agents facing a marginally binding constraint is large. This mass produces a jump in volatility as these agents cease to trade freely in the market and an associated jump down in the supply and demand for credit, resulting in a credit crunch. This credit crunch is smaller when mass is spread over several types and individual constraints will bind at different times, spreading the shock out over the state space. In this way preference heterogeneity dampens the effect of negative shocks on margins. This observation has implications for markets in which many diverse investors participate versus markets with a small number of participants, for example the market for index funds versus the market for more complex instruments such as derivative contracts. In addition, the model proposed can handle other types of convex portfolio constraints not treated here.

Convex portfolio constraints arise quite naturally in finance. A convex constraint simply states that the portfolio weights must lie in a convex set containing zero (see Stiglitz and Weiss (1981) for an example of microfoundations to credit constraints). In macroeconomics there are countless examples of particular models with market incompleteness which can be described in this setting of convex constraints, such as Aiyagari (1994); Kiyotaki and Moore (1997); Krusell and Smith (1998); Bernanke et al. (1999) and many others. This paper's approach could feasibly be applied to those settings.

This paper focuses on a particular application to margin constraints because of its tractability, ubiquity in financial markets, and use in the literature. A margin constraint essentially states that a borrower cannot borrow infinitely against their equity. This type of constraint is seen in consumer finance when borrowing money to purchase a home: one must almost always put up a down payment. In financial markets margin constraints arise in repo markets and other lending vehicles (see Hardouvelis and Peristiani (1992); Hardouvelis and Theodossiou (2002); Adrian and Shin (2010a) for empirical studies of margins). In fact real world experience motivated the theoretical study of leverage cycles initiated by Geanakoplos (1996). In addition limits to arbitrage and financial bubbles have been studied under margin constraints in the context of liquidity (see e.g. Brunnermeier and Pedersen (2009)). All of these types of phenomena will be studied in the present paper using a novel approach to solving general equilibrium incomplete financial markets.

General equilibrium in incomplete financial markets with heterogeneous power utility maximizers has been an open problem in mathematical finance. A fundamental paper by Cvitanić and Karatzas (1992) studied the general case of convex portfolio constraints in partial equilibrium. The authors developed an ingenious way to embed the agent in a series of fictitious economies, parameterized via a sort of Kuhn-Tucker condition, and then to select the appropriate market to make the agent just indifferent. However, their approach was to use convex duality to characterize the solution, which relies on a strict assumptions that the relative risk aversion be bounded above by one. This limitation led others to look to solve the primal problem directly, such as He and Pages (1993); Cuoco and He (1994); Cuoco (1997); Karatzas et al. (2003). These works use dense and complex mathematical techniques which may or may not provide tractable solutions for calculation. The present paper takes a more direct approach to solve the primal problem by noticing that homogeneous preferences are associated to a value function which factors into a function of wealth and a function of the state, under the appropriate ansatz. Using this ansatz, the Hamilton-Jacobi-Bellman equation becomes an ODE over a single state variable. Because of this it is possible to derive a system of equations governing consumption and stochastic discount factors as a function of the same state. This system resembles a Walrasian market and proof of existence of a solution follows identical steps to the classical demand system. Finally, it follows that the minimal state is simply the aggregate dividend, presenting not only a simple mathematical characterization of equilibrium, but also a solution which can feasibly be calculated using standard numerical methods.

1. The Cyclicality of Leverage

Beliefs driven leverage cycles are pro-cyclical according to theory. This implication is somewhat contradicted in several empirical studies, including Adrian and Shin (2010b). In that paper the authors note that the leverage cycle is only pro-cyclical for a particular sector of the economy, asset broker/dealers. However, those authors plot leverage as a function of total assets, which produces a mechanical correlation. Consider the definition of financial leverage:

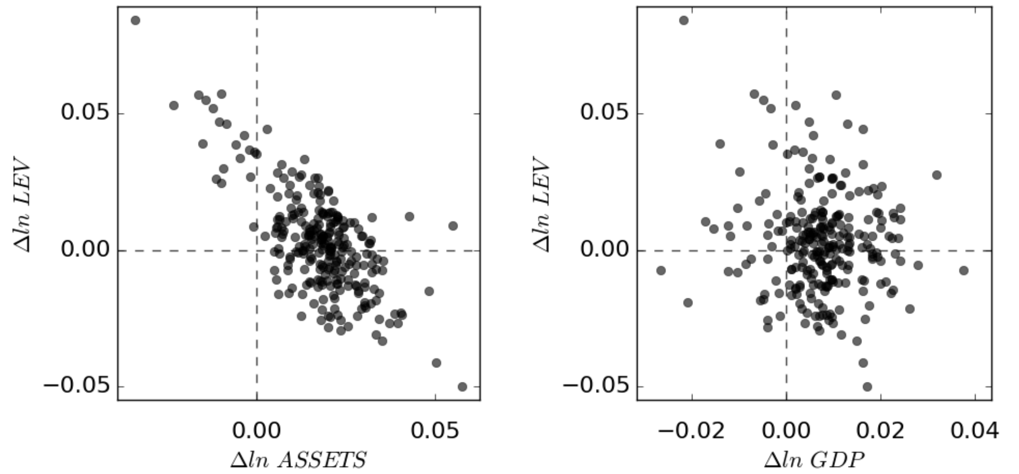
$$Leverage = \frac{Liabilities}{NetWealth} = \frac{Liabilities}{Assets - Liabilities}$$

Increases in balance sheet assets produce a negative correlation between leverage and assets[1](Ang et al. (2011)). Figure 1(a) plots the rate of growth in leverage against the rate of growth in assets for all sectors over 1952Q1 to 2017Q1 as measured from the US Flow of Funds. As you can see, there is a clear negative relationship.

Consider instead changes in GDP as a proxy for the business cycle. Figure 1(b) plots the rate of growth in leverage for all sectors against the rate of growth in GDP over the same period, again from U.S. Flow of Funds data. The previously clear negative relationship has disappeared, implying the leverage cycle is ambiguous in this sense. However, this ambiguity may simply be that there exists some other explanatory variable which drives the cyclicality of leverage, in particular preference heterogeneity.

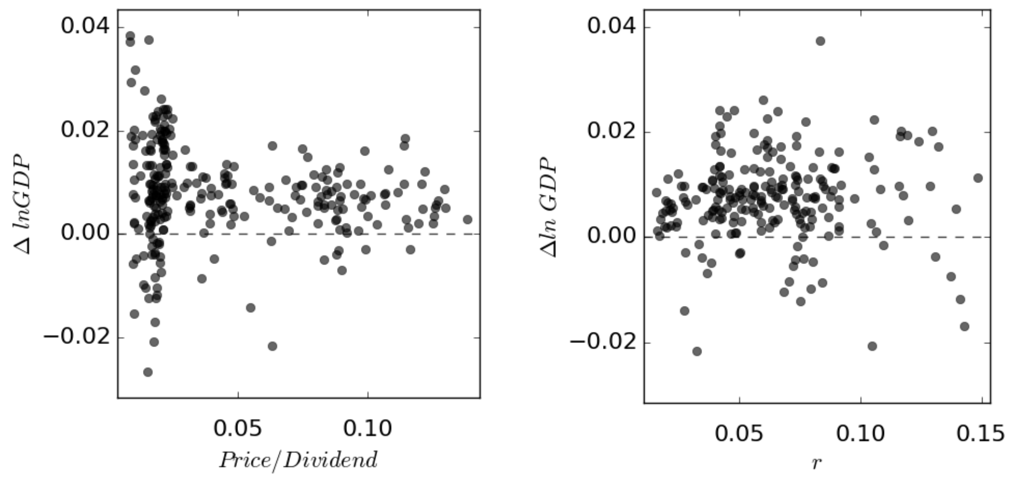
One proxy for preference heterogeneity is the price-dividend ratio. Asset prices will be high relative to dividends and vice-versa when the marginal agent in the economy is risk neutral. Figure 2(a) plots the growth rate in GDP against the price of S&P 500 divided by GDP (a measure of the price/dividend ratio of the total economy). Indeed we see that there is sub-

[1]However, this makes the fact that Adrian and Shin (2010b) find pro-cyclical leverage cycles for broker/dealers all the more substantial of a finding



(a) (b)

Fig. 1. Growth rate in leverage plotted against the growth rate in assets (Figure 1(a)) and againts the growth rate in GDP (Figure 1(b)) for all sectors. Source: FRB Flow of Funds Data.



(a) (b)

Fig. 2. Growth rate in GDP plotted against the price/dividend ratio (Figure 2(a)), proxied by the price of the S&P500 divided by GDP, and against the risk free rate (Figure 2(b)), proxied by the yield on constant maturity 10-year treasuries. Source: FRB Flow of Funds Data and FRED.

	Nonfinancial Corporations	Nonfinancial Private Business	HH's and Nonprofits	All Sectors
	(1)	(2)	(3)	(4)
Intercept	-0.0042	-0.0078	-0.0103	-0.0046
	(0.0618)	(0.0610)	(0.0599)	(0.0590)
$\Delta ln\ GDP$	0.1034	0.2117**	0.2827***	0.2397**
	(0.0994)	(0.0980)	(0.0963)	(0.0949)
S/D	-0.0045	-0.0119	-0.0209	-0.2107**
	(0.0874)	(0.0862)	(0.0846)	(0.0834)
$\Delta ln\ GDP * S/D$	-0.1727	-0.2726**	-0.3610***	-0.5157***
	(0.1070)	(0.1055)	(0.1036)	(0.1021)

Standard errors in parentheses.
* : $p \leq 0.1$, ** : $p \leq 0.05$, *** : $p \leq 0.01$

Table 1: Regression results for dependent variable $\Delta ln\ Lev$ for different sectors of the economy. A positive and significant coefficient on $\Delta ln\ GDP$ implies procyclicality, while a negative and significant coefficient on the interaction with S/D implies counter-cyclicality when the price dividend ratio is high. Note: Variables are normalized using z-score.

stantial dispersion in this measure. The price/dividend ratio bunches towards the origin as asset prices have been rising over time, but there is little evidence for a clear positive or negative relationship with GDP growth. For this reason, we can consider the correlations between these variables, captured by the following regression:

$$\Delta ln\ Lev = \alpha + \beta_1 \Delta ln\ GDP + \beta_2 \Delta ln\ GDP * \frac{S}{D} + \beta_3 \frac{S}{D}$$

The cyclicality of the leverage cycle is then captured by the slope with respect to the growth rate in GDP, that is

$$\partial_{\Delta ln\ GDP} \Delta ln\ Lev = \beta_1 + \beta_2 \frac{S}{D}$$

The leverage cycle is pro- or counter-cyclical as this value is positive or negative, respictively. Table 1 reports the results for several specifications, studying different subsamples of the economy.

The results imply that the cyclicality of leverage is not the same for all values of the price-dividend ratio. Column 4 gives results for all sectors included in the US Flow of Funds. Leverage growth is positively correlated with GDP growth when the price dividend ratio is low. As asset prices rise the effect changes sign and the correlation becomes negative. Changes in the price-dividend ratio imply changes in the preferences of the marginal agent pricing risky assets. When the price-dividend ratio is low the marginal agent is risk averse, while when the price-dividend ratio is high the marginal agent is more risk neutral. Thus the leverage cycle is pro-cyclical when risk-averse agents dominate and counter-cyclical when risk-neutral agents dominate.

This result is fairly robust to other measures of marginal preferences. One problem could be the heteroscedasticity exhibited by GDP growth over the price/dividend ratio in Figure 2(a). Consider the risk free rate as a proxy for the marginal agent, which is plotted in Figure 2(b) against GDP growth. In this case the dispersion of GDP growth is more uniform over values of the interest rate. Define a similar set of regressions as before, i.e.:

$$\Delta ln\ Lev = \alpha + \beta_1 \Delta ln\ GDP + \beta_2 \Delta ln\ GDP * r + \beta_3 r$$

Again the cyclicality is captured by the slope with respect to the growth rate in GDP:

$$\partial_{\Delta ln\ GDP} \Delta ln\ Lev = \beta_1 + \beta_2 r$$

In this case we should expect the sign to flip. The interest rate is high when the marginal agent is risk-averse and low when the marginal agent is risk-neutral. Table 2 reports the results. Leverage growth comoves positively with GDP growth and the interest rate is high and negatively when the interest rate is low. This result again implies that the cyclicality of the leverage cycle depends in the same way as before on the preferences of the marginal agent.

The regression results highlight how the cyclicality of the leverage cycle relates to financial variables and, in turn, preferences. Agents are likely to be constrained when asset prices are low, producing a pro-cyclical leverage cycle.

	Nonfinancial Corporations	Nonfinancial Private Business	HH's and Nonprofits	All Sectors
	(1)	(2)	(3)	(4)
Intercept	0.0170	0.0838	0.0948	0.0227
	(0.0674)	(0.0638)	(0.0674)	(0.0672)
$\Delta ln\ GDP$	−0.5144***	−0.4412**	−0.5330***	−0.9689***
	(0.1907)	(0.1808)	(0.1908)	(0.1904)
r	0.0164	−0.0847	0.0018	0.0549
	(0.0789)	(0.0748)	(0.0790)	(0.0788)
$\Delta ln\ GDP * r$	0.4121**	0.4095**	0.5131***	0.7603***
	(0.1700)	(0.1611)	(0.1701)	(0.1697)

Standard errors in parentheses.
$^{*}: p \leq 0.1$, $^{**}: p \leq 0.05$, $^{***}: p \leq 0.01$

Table 2: Regression results for dependent variable $\Delta ln\ Lev$ for different sectors of the economy. A negative and significant coefficient on $\Delta ln\ GDP$ implies counter-cyclicality, while a positive and significant coefficient on the interaction with r implies pro-cyclicality when the interest rate is high. As opposed to Table 1, r is high when the risk averse agent dominates, exactly when the price-dividend ratio is low. Note: Variables are normalized using z-score.

Agents will be far from their constraint when asset prices are high, producing a counter-cyclical the leverage cycle. This paper proposes to explain this phenomenon by preference heterogeneity. Asset prices are low or high relative to dividends depending on whether the marginal agent is more or less risk averse. The remainder of the paper proposes a model for this situation and solves for the equilibrium.

2. A Model of Preference Heterogeneity

Consider a continuous time, infinite horizon Lucas (1978) economy with one consumption good. This consumption good, denoted D_t, is produced by a per capita tree[2], whose dividend follows a geometric Brownian motion (GBM):

$$\frac{dD_t}{D_t} = \mu_D dt + \sigma_D dW_t$$

where W_t is a standard Brownian motion and (μ_D, σ_D) are constants.

The economy is populated by an arbitrary number N of atomistic agents indexed by $i \in \{1, \ldots, N\}$. Agents have constant relative risk aversion (CRRA) preferences and differ in their rate of relative risk aversion and initial wealth, x_i.

Investors have access to financial markets where they can continuously trade in a stock and bond. The bond and stock prices, denoted S_t^0 and S_t respectively, are assumed to follow a GBM and an exponential process, respectively:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t \qquad (1) \qquad\qquad \frac{dS_t^0}{S_t^0} = r_t dt \qquad (2)$$

where (μ_t, σ_t, r_t) are determined in equilibrium. Denote by X_{it} an individual's

[2]One can either consider N identical trees such that $D_0 = 1.0$ (for example) or one big tree where $D_0 = N$, the two cases are equivalent.

wealth at time t and by π_{it} the share of an individual's wealth invested in the risky stock, which implies $1 - \pi_{it}$ is the share invested in the bond.

2.1. Budget Constraints and Individual Optimization

Individual investors solve a utility maximization problem subject to a self-financing budget constraint and a portfolio constraint:

$$\begin{aligned} \max_{\{c_{it},\pi_{it}\}_{t=0}^{\infty}} \quad & \mathbb{E}\int_0^{\infty} e^{-\rho t}\frac{c_{it}^{1-\gamma_i}}{1-\gamma_i}dt \\ \text{s.t.} \quad & dX_{it} = \left[X_{it}\left(r_t + \pi_{it}(\mu_t + \frac{D_t}{S_t} - r_t)\right) - c_{it}\right]dt + X_{it}\pi_{it}\sigma_t dW_t \\ & \pi_{it} \in \Pi_i \end{aligned}$$

where $\Pi_i \subseteq \mathbb{R}$ represents a closed, convex region of the portfolio space which contains $\{0\}$. For example $\Pi_i = \mathbb{R}$ is the unconstrained case, $\Pi_i = \mathbb{R}^+$ is a short sale constraint, $\Pi_i = \{\pi : \pi \leq m \mid m \geq 0\}$ is a margin constraint. This set is allowed to differ across agents, as implied by the subscript. This paper focuses on an application to margin constraints, but the approach is applicable to any constraint which can be written as a function of the aggregate state[3].

2.2. Equilibrium

Investors are considered to be atomistic and thus I consider an Arrow-Debreu type equilibrium, where individuals do not consider their own impact on prices.

Definition 1. *An equilibrium in this economy is defined by a set of processes* $\{r_t, S_t, \{c_{it}, X_{it}, \pi_{it}\}_{i=1}^N\}\ \forall\ t$*, given preferences and initial endowments, such that* $\{c_{it}, X_{it}, \pi_{it}\}$ *solve the agents' individual optimization problems and the*

[3] An important limitation of the approach presented here is that it cannot treat constraints which depend explicitly on individual wealth. The reason why will become apparent later, but revolves around the necessity to write equilibrium objects as functions of aggregate variables only.

following set of market clearing conditions is satisfied:

$$\frac{1}{N}\sum_i c_{it} = D_t \ , \ \frac{1}{N}\sum_i (1-\pi_{it})X_{it} = 0 \ , \ \frac{1}{N}\sum_i \pi_{it}X_{it} = S_t \tag{3}$$

I study Markovian equilibria such that the problem can be written as a function of a single state variable. This is one of the key mathematical insights of the paper and has implications for the study of general equilibrium models under constraints.

3. Equilibrium Characterization

To solve this problem I begin with the approach of Cvitanić and Karatzas (1992). This method uses a fictitious, unconstrained economy and a shadow cost of constraint, or Lagrange multiplier, to find the correct pricing process. Unlike in their work I do not use a duality approach, but show how the primal problem can be written in terms of a single state variable. This approach works because of the homogeneity of CRRA preferences and will likewise work for any homogeneous utility function, including Epstein-Zin[4]. The reason this works is because when utility functions are functionally homogeneous an individual's consumption choice is linear in their wealth and, in turn, their portfolio choice is independent of wealth. In this case the value function factors and the resulting ODE is no longer a function of individual wealth. Wealth-consumption ratios become the key objects and we can use them in market clearing to derive the solution. This process will be described in the following subsections.

[4] In particular, first order conditions from a dynamic program give consumption as $c = u'^{-1}\left(\partial_X J(X,Y)\right)$, where Y is any arbitrary, aggregate state vector and X an individual's wealth. We would like to find $c = X/V(Y)$. Equate these and rearrange to find $\partial_X J(X,Y) = u'(X/V(Y))$. When preferences are homogeneous of degree $k+1$, $u'(\cdot)$ is homogeneous of degree k. Thus $\partial_X J(X,Y) = u'(1)V(Y)^{-k}X^k$. By integrating with respect to X one finds a proposal for the value function such that consumption is a linear function of wealth.

3.1. Optimality in Fictitious Unconstrained Economy

In order to find the constrained equilibrium, we define new processes for individual prices, which are "adjusted" by a process ν_{it}, considered the shadow cost of constraint:

$$\frac{dS_t^0}{S_t^0} = (r_t + \delta_i(\nu_{it}))dt$$
$$\frac{dS_t}{S_t} = (\mu_t + \nu_{it} + \delta_i(\nu_{it}))dt + \sigma_t dW_t$$

The function $\delta_i(\cdot)$ is the support function of Π_i, which is defined as

$$\delta_i(\nu) = \sup_{\pi \in \Pi_i} (-\nu\pi)$$

In addition, this gives rise to the effective domain of ν_{it} defined by $\mathcal{N}_i = \{\nu \in \mathbb{R} : \delta_i(\nu) < \infty\}$. Each agent solves their optimization problem in the face of their individual, fictitious financial market. This gives rise to a large number of state price densities for each individual, depending on how tight is their constraint.

Define the stochastic discount factor (SDF) of an individual agent as a GBM which evolves as a function of the individual's adjustment:

$$\frac{dH_{it}}{H_{it}} = -(r_t + \delta_i(\nu_{it}))dt - \left(\theta_t + \frac{\nu_{it}}{\sigma_t} m\right) dW_t \tag{4}$$

By a straight-forward application of the martingale method (Karatzas et al. (1987)) in this fictitious economy one finds individual consumption as a function of individual SDF's:

$$c_{it} = \left(\Lambda_i e^{\rho t} H_{it}\right)^{-\frac{1}{\gamma_i}} \tag{5}$$

for all i, where Λ_i is the Lagrange multiplier associated to the static budget constraint. In complete markets, i.e. the case where $\Pi_i = \mathbb{R}\ \forall\ i$, the SDF's coincide and the ratios of marginal utilities are constant. However, when agents are constrained in their portfolio choice this is not the case and we have

$$\frac{c_{it}^{-\gamma_i}}{c_{jt}^{-\gamma_j}} = \frac{\Lambda_i H_{it}}{\Lambda_j H_{jt}}$$

However, we know how to solve this problem in terms of D_t in complete markets (Abbot (2017)). Intuitively, if we know the portfolio weight an agent would choose in complete markets and we know the constraint set, there should be nothing to stop us from finding their portfolio choice in the constrained setting. This will be the case, but will require constructing consumption weights, a non-trivial task given ratios of marginal utilities are not necessarily constant.

3.2. From Partial Equilibrium to Market Clearing

The above results represent partial equilibrium of individual agents. To consider general equilibrium we must aggregate these results across agents, but we have a large number of unknowns, as each individual faces a different SDF. Nevertheless, we can use knowledge about the complete markets solution and the assumption of CRRA preferences to characterize the equilibrium. In particular, we know that in complete markets under CRRA preferences (or any preferences which are homogeneous of some degree) that the value function factors into a function of wealth and a function of dividends. In this spirit, assume an agent's wealth-consumption ratio $V_i(D) = X_{it}/c_{it}$ is a function of the dividend and substitute into the market clearing conditions in Eq. (3) to derive the following proposition:

Proposition 1. *Assuming the wealth-consumption ratio of an individual agent is given by $V_i(D) = X_{it}/c_{it}$, individual consumption weights $c_{it}/D_t = \omega_{it}$ and stochastic discount factors H_{it} satisfy a system of coupled non-linear* *equations given by*

$$1 = \frac{1}{N}\sum_j \left(\frac{\Lambda_j H_{jt}}{\Lambda_i H_{it}}\right)^{-\frac{1}{\gamma_j}} \omega_{it}^{\frac{\gamma_i}{\gamma_j}} D^{\frac{\gamma_i}{\gamma_j}-1} \tag{6}$$

$$0 = \frac{1}{N}\sum_j (1-\pi_{jt})V_j(D) \left(\frac{\Lambda_j H_{jt}}{\Lambda_i H_{it}}\right)^{-\frac{1}{\gamma_j}} \omega_{it}^{\frac{\gamma_i}{\gamma_j}} D^{\frac{\gamma_i}{\gamma_j}} \tag{7}$$

for all i *in* $\{1, ..., N\}$*.*

Proposition 1 implies several things about the solution. Eqs. (6) and (7) are a system of $2N$ equations in $2N$ unknowns. This is a promising fact for solving the model, but how the SDF's enter the equations is problematic. The solution to this system of equations will not be unique since the SDF's enter as ratios. Given a solution, scaling all of the SDF's by the same constant also produces a solution. However, the derivation comes from an application of Walras' law, which reminds us that this represents a price system[5], where H_{it} is an individual's price for consumption. Although it may not be possible to characterize these prices explicitly over the state space, the relative prices of two individuals is pinned down as a function of D. In addition, it is possible to use the standard approach to proving existence of equilibrium in a pricing system to show that the above system has at least one solution.

Proposition 2. *The system of equations Eqs.* (6) *and* (7) *admits at least one solution.*

In addition to this system, we can note that the ratio of SDF's $h_{ijt} = H_{it}/H_{jt}$, is Markov and satisfies a GBM. An application of Itô's lemma gives the dynamics as

$$\frac{dh_{ijt}}{h_{ijt}} = \left[\delta_j(\nu_{jt}) - \delta_i(\nu_{it}) + \left(\theta_t + \frac{\nu_{it}}{\sigma_t}\right)\left(\frac{\nu_{jt}-\nu_{it}}{\sigma_t}\right)\right] dt + \frac{\nu_{jt}-\nu_{it}}{\sigma_t} dW_t \tag{8}$$

There remains only a single dimension of risk and all of the heterogeneity is static, which implies the key variables to find are the adjustments ν_{it}. One can see that when two agents are not constrained, i.e. $\nu_{it} = \nu_{jt} = 0$, the

[5] I must thank Gordon Zitkovic for pointing out this fact.

ratio has no drift or diffusion and remains constant, as in complete markets. When one or both agents is constrained the ratio will move depending on how tight is the constraint and the sign of ν_{it} and $\delta_i(\nu_{it})$. Although the individual stochastic discount factor may not be a function only of a low dimensional state, the relative SDF's will depend on how the constraints bind and interact with the state variable.

Given this characterization of the consumption weights and discount factors in terms of a single state variable, there is only a single variable agents need in order to solve their static optimization problem and to understand the optimization of all other agents. This points towards a Hamilton-Jacobi-Bellman (HJB) representation of the value function via the Bellman principle in terms of a small number of state variables. This is interesting from a mathematical standpoint given its relationship to the theory of mean field games (MFG) with common noise. Typically such problems require the use of infinite dimensional stochastic processes (Bensoussan et al. (2015); Carmona et al. (2014) and others) and have often been treated in the economics literature by the use of simulations (Krusell and Smith (1998)) or linearizations in high dimension (Ahn et al. (2016)). Here, however, there is only a single state process and the dimension remains finite as the number of agents grows. This points towards a new way to study MFG models in financial markets when the preferences are homogeneous of any degree.

3.3. General Equilibrium Characterization

Equilibrium is characterized by first assuming the existence of a Markovian equilibrium, then recovering the adjustments ν_{it} using the Kuhn-Tucker conditions, and finally deriving a system of ODE's for wealth-consumptions ratios. Given this it is possible to prove optimality of the value functions.

Proposition 3. *The interest rate and market price of risk can be shown to be functions of weighted averages of individuals' consumption weights, pref-*

erence parameters, and adjustments such that

$$\theta_t = \frac{N}{\sum_i \frac{\omega_{it}}{\gamma_i}} \left(\sigma_D - \frac{1}{\sigma_t N} \sum_i \frac{\omega_{it}\nu_{it}}{\gamma_i} \right) \tag{9}$$

$$r_t = \frac{N}{\sum_i \frac{\omega_{it}}{\gamma_i}} \left(\mu_D + \frac{\rho}{N} \sum_i \frac{\omega_{it}}{\gamma_i} - \frac{1}{N} \sum_i \frac{\omega_{it}}{\gamma_i} \delta_i(\nu_{it}) \right. \tag{10}$$

$$\left. - \frac{1}{2N} \sum_i \frac{1+\gamma_i}{\gamma_i^2} \left(\theta_t + \frac{\nu_{it}}{\sigma_t} \right)^2 \omega_{it} \right) \tag{11}$$

Consumption weights follow a geometric Brownian motion whose dynamics are given by:

$$\frac{d\omega_{it}}{\omega_{it}} = \mu_{\omega it} dt + \sigma_{\omega it} dW_t$$

where

$$\mu_{\omega it} = \frac{1}{\gamma_i} \left(r_t + \delta_i(\nu_{it}) - \rho + \frac{1}{2} \frac{1+\gamma_i}{\gamma_i} \left(\theta_t + \frac{\nu_{it}}{\sigma_t} \right)^2 - \left(\theta_t + \frac{\nu_{it}}{\sigma_t} \right) \right)$$

$$+ \sigma_D - \mu_D \tag{12}$$

$$\sigma_{\omega it} = \frac{1}{\gamma_i} \left(\theta_t + \frac{\nu_{it}}{\sigma_t} \right) - \sigma_D \tag{13}$$

The interest rate and market price of risk take a typical form, but are augmented by the adjustment to individuals' marginal utilities. First notice that the market price of risk is determined by the fundamental volatility σ_D divided by the weighted average of elasticity of intertemporal substitution (EIS), exactly as in complete markets (Abbot (2017)). In addition the constraint will either increase or reduce the market price of risk, depending on the domain of ν_{it}. In the case of margin constraints $\nu_{it} \leq 0$, so the market price of risk will be weakly higher under constraint. The magnitude of deviation from complete markets depends on several factors: the difference will be greater in times of low volatility when risk neutral agents are constrained than in times of high volatility when risk averse agents are constrained. When one group is constrained, unconstrained agents price the risky asset and the price will necessarily be higher or lower if the marginal agent is less or more risk averse, respectively. However, the effect of volatility implies that in times when stock price volatility is low, greater constraint implies greater returns. This correlation is driven by the fact that agents cannot borrow to take advantage of the returns, producing the type of liquidity effect described in the limits-to-arbitrage literature (e.g. Brunnermeier and Pedersen (2009)). Risk neutral agents would arbitrage away the high returns, but cannot because of their margin constraint. This creates room for bubbles to form and for the asset price to be greater than the fundamental price.

The interest rate similarly exhibits a familiar shape. We see an intertemporal transfer term and a prudence term, but these are augmented by the constraint. Under a homogeneous margin constraint, $\delta_i(\nu_{it}) = -m\nu_{it}$, but recall that $\nu_{it} \leq 0$. Constrained agents are unable to supply bonds to the market in order to transfer consumption and wealth from the future to today. A lower supply of bonds pushes up the price and down the interest rate. At the same time constraint affects the prudence motive by changing the relative prices of bonds and stocks. When some agents are constrained, the market price of risk is higher and individuals will demand fewer bonds. Lower demand pushes the price down and the interest rate up, counteracting the intertemporal effect. Together these forces produce an equity risk premium which depends on the shape of heterogeneity, the degree of constraint, and the state variable, all driven by the individual consumption weights which determine the marginal agents.

Since θ_t and r_t are functions of $\{\omega_{it}\}_{i=1}^N$ and since these are functions of D_t, it remains to show that $\{\nu_{it}\}_{i=1}^N$ is as well in order to show that an equilibrium with only D_t as a state variable is appropriate. First, one can derive a system of ODE's for individual wealth/consumption ratios, from which one can determine the adjustments.

Proposition 4. *Assuming there exists a Markovian equilibrium in D_t, the individuals' wealth-consumption ratios, given by $V_i(D)$ satisfy ODE's given* *by*

$$
\begin{aligned}
0 =& 1 + \frac{\sigma_D^2 D^2}{2} V_i''(D) + \left[\frac{1-\gamma_i}{\gamma_i}\left(\theta_t + \frac{\nu_{it}}{\sigma_t}\right)\sigma_D + \mu_D\right] D V_i'(D) \\
&+ \left[(1-\gamma_i)(r_t - \delta_i(\nu_{it})) - \rho + \frac{1-\gamma_i}{2\gamma_i}\left(\theta_t + \frac{\nu_{it}}{\sigma_t}\right)^2\right]\frac{V_i(D)}{\gamma_i}
\end{aligned}
\tag{14}
$$

which satisfy boundary conditions

$$
\lim_{D \to D^*} V_i(D) = \frac{\gamma_i}{\rho - (1-\gamma_i)\left(\frac{\theta(D^*)^2}{2\gamma_i} + r(D^*)\right)} \quad \textit{for } D^* \in \{0, \infty\} \tag{15}
$$

Portfolios are given as functions of $V_i(D)$*:*

$$
\pi_{it} = \frac{1}{\gamma_i \sigma_t}\left(\gamma_i \sigma_D D \frac{V_i'(D)}{V_i(D)} + \theta_t + \frac{\nu_{it}}{\sigma_t}\right) \tag{16}
$$

And stock price volatility is given by

$$
\sigma_t = \sigma_D \left(1 + D\frac{\mathcal{S}'(D)}{\mathcal{S}(D)}\right) \tag{17}
$$

where

$$
\mathcal{S}(D) = \frac{1}{N}\sum_i V_i(D)\omega_{it} \tag{18}
$$

The constraint affects individual choice through their wealth-consumption ratio and their portfolio weight, and thus indirectly determines asset pricing variables. Constraint does not directly affect the diffusion term (the second derivative), but only the convection and source terms (the first derivative and the undifferentiated term). Constraint changes how quickly an agent is able to accumulate wealth through leverage and how much they are exposed to the aggregate dividend. This exposure comes through their portfolio choice (Eq. (16)), which exhibits the typical Merton (1971) myopic and hedging terms, but also a constraint term. Interestingly, this constraint term is scaled down by γ_i, implying that the amount of adjustment differs across individuals depending on preferences. In order to compensate a risk averse agent for being constrained their adjustment will be lower than the equivalent compensation for a risk neutral agent. In this way, constraining risk neutral agents will have a greater effect on equilibrium than constraining risk averse agents. Finally, excess volatility arises in Eq. (17) driven by the slope of the wealth consumption ratio. However, as mentioned before, constraint affects this slope through the convection term and (in particular under margin constraints) will reduce excess volatility. This reduction is driven by constraint essentially reducing heterogeneity. In the extreme if all agents were constrained there would be no trade and the volatility would reduce to the autarkical case, that is to the fundamental volatility.

The functional form of the adjustments ν_{it} depends on the type of constraint. To that end, I will focus from here only on margin constraints. From an applied perspective, this is most likely the constraint affecting individuals most often, for example when borrowing to purchase a home, attend college, start a business, etc., a loan with no margin is essentially non-existent. The following proposition gives the functional form for the adjustments under homogeneous margin constraints when $\pi_{it} \leq m$ for all i, where $m \geq 0$, which implies an effective domain of $\mathcal{N}_i = \{\nu : \nu \leq 0\}$.

Proposition 5. *Under margin constraints, adjustments* $\{\nu_{it}\}_{i=1}^{N}$ *satisfy a system of* N *equations:*

$$\nu_{it} = \min\left\{0; m\gamma_i\sigma_t^2\left(1 - \frac{1}{m\sigma_t}\left(\frac{\theta_t}{\gamma_i} + \sigma_D D_t \frac{V_i'(D)}{V_i(D)}\right)\right)\right\} \tag{19}$$

The equations in Proposition 5 essentially close the model. Given the above propositions, it is possible to prove that this is indeed an equilibrium.

Proposition 6. *Suppose there exist bounded positive functions* $V_i(D) \in C^1[0,\infty) \cup C^2[0,\infty)$ *that satisfy the system of ODE's in Eq.* (14) *and boundary conditions Eq.* (15)*. Additionally, assume the processes* θ_t*,* ν_{it}/σ_t*,* σ_t*, and* π_{it} *are bounded and that* $|\sigma_t| > 0$*. Then there exists a Markovian equilibrium satisfied by Propostions 3 to 5*

This is not to say that this is the only equilibrium, as one could characterize a higher dimensional equilibrium taking consumption weights as state variables, as is done in Chabakauri (2015) or Gârleanu and Panageas (2015) for two agents. However, what this equilibrium shows is that it is possible to solve the model in a lower dimensional space. Because all of the risk in the economy is driven by a single Brownian motion and because of the simple functional form provided by CRRA preferences, we do not need to resort to higher dimensional methods and this equilibrium should hold even in the continuum, as $N \to \infty$.

4. Numerical Solution

This section presents numerical results for several assumptions about the distribution of preferences. First, the case of two types is evaluated and the cyclicality of the leverage cycle is emphasized. The leverage cycle is pro- or counter-cyclical depending on the marginal agent. In addition, the severity of cycles dpends on whether the risk neutral agent is constrained. This solution is discussed in relation to previous solutions with two types, emphasizing the difference created by the choice of state variable. Second, I study increasing the number of types over a given support is and find that heterogeneity reduces the severity of deviations from complete markets for most financial variables, despite leverage and the market price of risk remaining high. This observation implies that the number of types is important when considering the effect of margin constraints as a policy tool. Over all simulations I hold fixed $(\mu_D, \sigma_D, \rho) = (0.01, 0.032, 0.02)$, chosen to compare to Chabakauri (2015).

4.1. Two Types and Leverage Cycles

Consider the case of two agents with relative risk aversion $(\gamma_1, \gamma_2) = (1.1, 3.0)$ facing a margin constraint of $m = 1.25$. This corresponds to a constraint on individuals' financial leverage of $Debt/Equity = |1 - \pi_{it}|/\pi_{it} \leq 0.2$. This constraint produces substantial deviations from complete markets, a jump in volatility, a shifting out of the inflection point in leverage, and a singularity producing a jump in the leverage cycle.

Consider first financial variables. Figure 3 plots financial variables in levels for complete and incomplete markets. In both cases the risk averse agent dominates as dividends fall, pushing up asset prices, the interest rate, and the market price of risk. Lower values of the dividend also correspond to times of greater trade between the two types, increasing volatility. It is also clear where the constraint begins to bind for the risk neutral agent who wishes to lever up more in bad times, but cannot. The constraint indirectly forces the risk averse agent to hold a greater share of the risky asset. Returns are greater and asset prices higher to compensate these risk averse agents. In addition the reduction in trade associated with the constraint causes volatility to jump down.

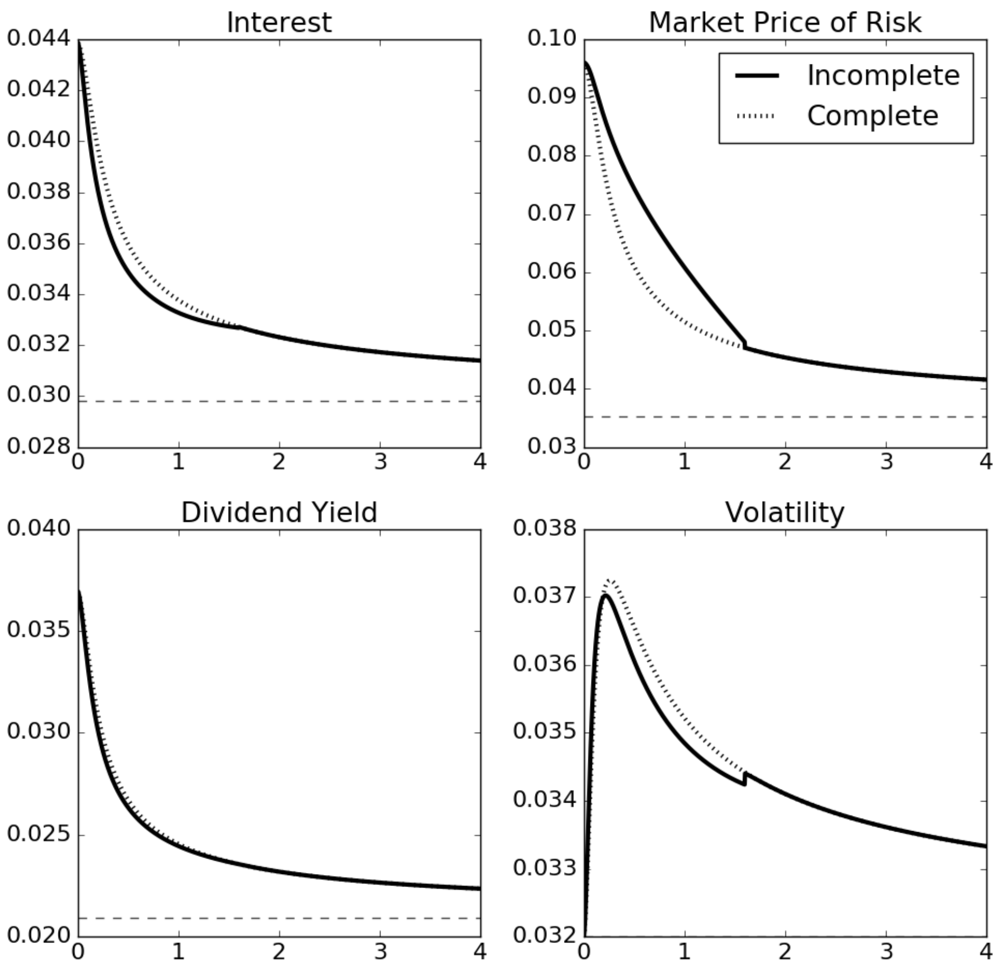


Fig. 3. Financial variables under margin constraint for 2 types when $\gamma_1 = 1.1$ and $\gamma_2 = 3.0$.

These effects are more clearly seen in deviations from complete markets (i.e. $(Y_I/Y_c - 1) \times 100\%$), shown in Figure 4. Returns are over 20% higher in the constrained region to compensate risk averse agents for holding a larger share of the risky asset. One would think this would push down asset prices, as risk averse agents have a lower autarky price for risky assets, but in fact the dividend yield is lower (and the price/dividend ratio thus higher) under constraint. Risk neutral agents are limited by their constraint from participating and arbitraging away the high returns. This limit to arbitrage is similar to the type of liquidity constraints which are posited in other settings (e.g. Brunnermeier and Pedersen (2009)). Asset prices are above their fundamental value because of this limit to risk neutral agents' ability to profit form the arbitrage and a financial bubble arises (as in Hugonnier (2012)). Despite this bubble and high asset prices, risk averse agents see the jump down in volatility and the increase in the market price of risk and shift wealth to the risky asset. This produces a contraction in the supply of credit which pushes down the interest rate and affects leverage.

The effect on leverage is substantial given both a supply effect and a demand effect. Figure 5 plots several measures of leverage and the leverage cycle. The demand for credit is artificially lower under constraint when risk neutral agents cannot leverage up. The supply of credit is also reduced. A jump down in volatility produces a jump up in expected returns on risky assets. Risk averse agents shift wealth into risky shares and the supply of credit contracts. This produces an ambiguous effect on the interest rate at the boundary when constraint binds. However, there is an unambiguous change in the cyclicality of leverage.

Leverage cycles are both pro- and counter-cyclical in both complete and incomplete markets, but the dynamics of this cyclicality is vastly different under the two regimes. In complete markets, the slope of leverage varies smoothly, moving from positive to negative as one moves through the state space. Only in very bad states does leverage exhibit pro-cyclicality, as risk averse agents begin to dominate and the interest rate becomes too high for risk neutral agents to desire to borrow. This inflection point becomes a singularity under margin constraints. In Figure 5(b) we can see a jump at

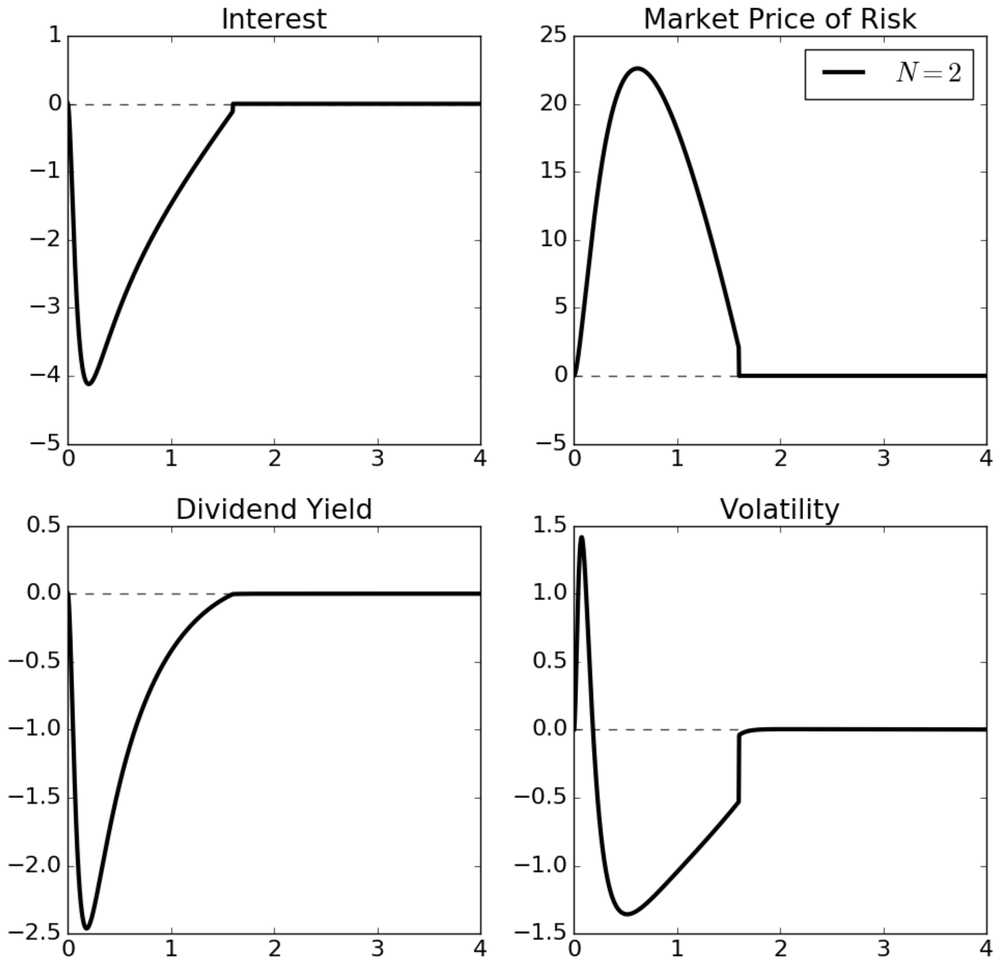
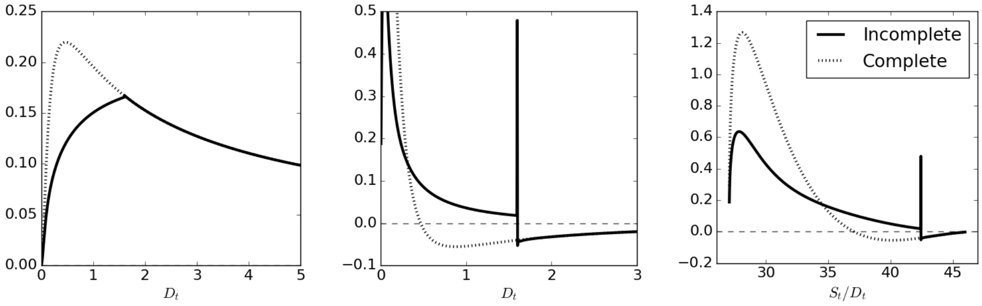


Fig. 4. Financial variables in percent deviations from complete markets ($100 \times (Y_I - Y_c)/Y_c$) under margin constraint for 2 types when $\gamma_1 = 1.1$ and $\gamma_2 = 3.0$.

the interface between the constrained and the unconstrained region. At the interface a small negative shock causes the economy to jump from counter- to pro-cyclical leverage cycles because of the formation of a bubble and the artificial limit on borrowing. In Figure 5(c) we see that the constraint causes this singularity to shift out substantially over values of the price/dividend ratio. Although only a small portion of the state space is constrained, the constraint is in effect for a very large portion of the range of asset prices achieved by the economy.

Typically one considers two agents and takes the consumption shares



(a) (b) (c)

Fig. 5. Leverage for two agents when $(\gamma_1, \gamma_2) = (1.1, 3.0)$. Figure 5(a) plots aggregate financial leverage, Figures 5(b) and 5(c) plots $\partial_D\ Lev$ as a function of D_t and S_t/D_t, respectively.

of one individual as the state variable when studying with heterogeneous preferences (e.g. Chabakauri (2013, 2015); Gârleanu and Panageas (2015) etc.). This is convenient and intuitive from a modeling perspective and gives results which imply binding constraints over a substantial portion of the state space. However, this share is not readily observable in economic data. The equilibrium presented above is in terms of an observable quantity and the numerical solution shows that the constrained portion of the state space is actually rather small. In Figure 6 we can see that the effect of constraint on consumption weights is small for two types. Contrast this to if we were considering consumption weights themselves as a state variable, in which case it would seem that more than half of the state space is constrained. We see that the constraint binds only in very bad times when characterizing the equilibrium over D_t, .

4.2. Increasing Heterogeneity and The Severity of Crisis

In order to consider what happens when the degree of heterogeneity increases, consider simulating 2, 5, 10, and 20 agents, but now assume their preferences are evenly spaced over $[1.1, 3.0]$. Increasing the number of preference types has a surprising effect on the severity of crises and the amount to which the constraint affects market outcomes.

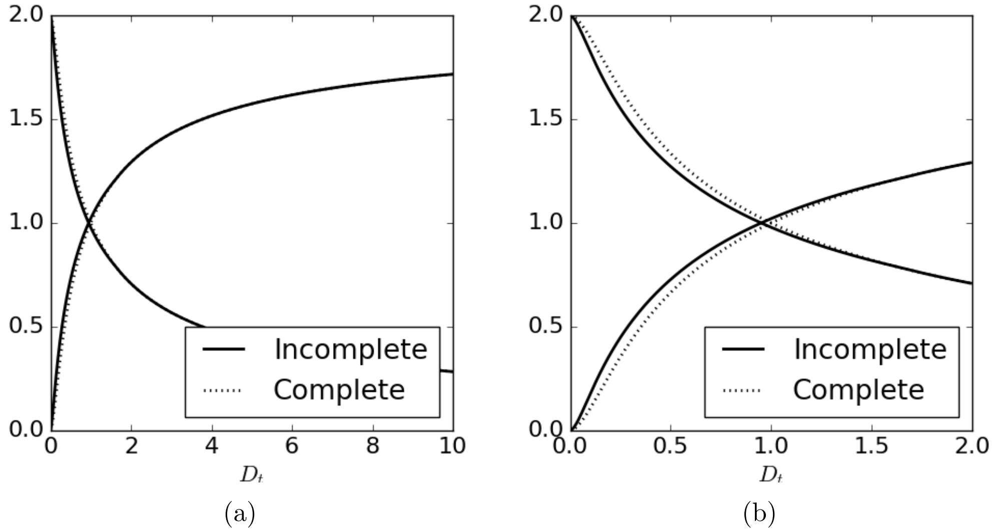


Fig. 6. Consumption weights (ω_1, ω_2) for two types when $(\gamma_1, \gamma_2) = (1.1, 3.0)$, over a broader (Figure 6(a)) and tighter (Figure 6(b)) portion of the state space.

In Figure 7 you can see relative deviations of financial variables from the complete markets case. First, with intermediate preference types there is more liquidity in the market, increasing the market-clearing level of portfolio weights. This increase causes the constraint to bind for every agent at a higher level of the dividend, causing the onset of crisis (considered to be the constrained region) to be earlier. However, the deviation is smaller for most variables. This is because each individual agent has a smaller weight in the economy and their constraint matters less. At the same time, the market price of risk remains high. As before, the returns on risky assets must be higher to compensate risk averse agents for holding a larger share, but by spreading agents out over the support of preferences, the severity of crisis is reduced. In this way, preference heterogeneity and constraint generate a limit to arbitrage. There exists excess returns from which risk neutral agents are unable to profit because of margin constraints.

In addition, the cyclicality of leverage is affected by the number of types. Figure 8 presents the aggregate leverage in each simulation and Figure 9

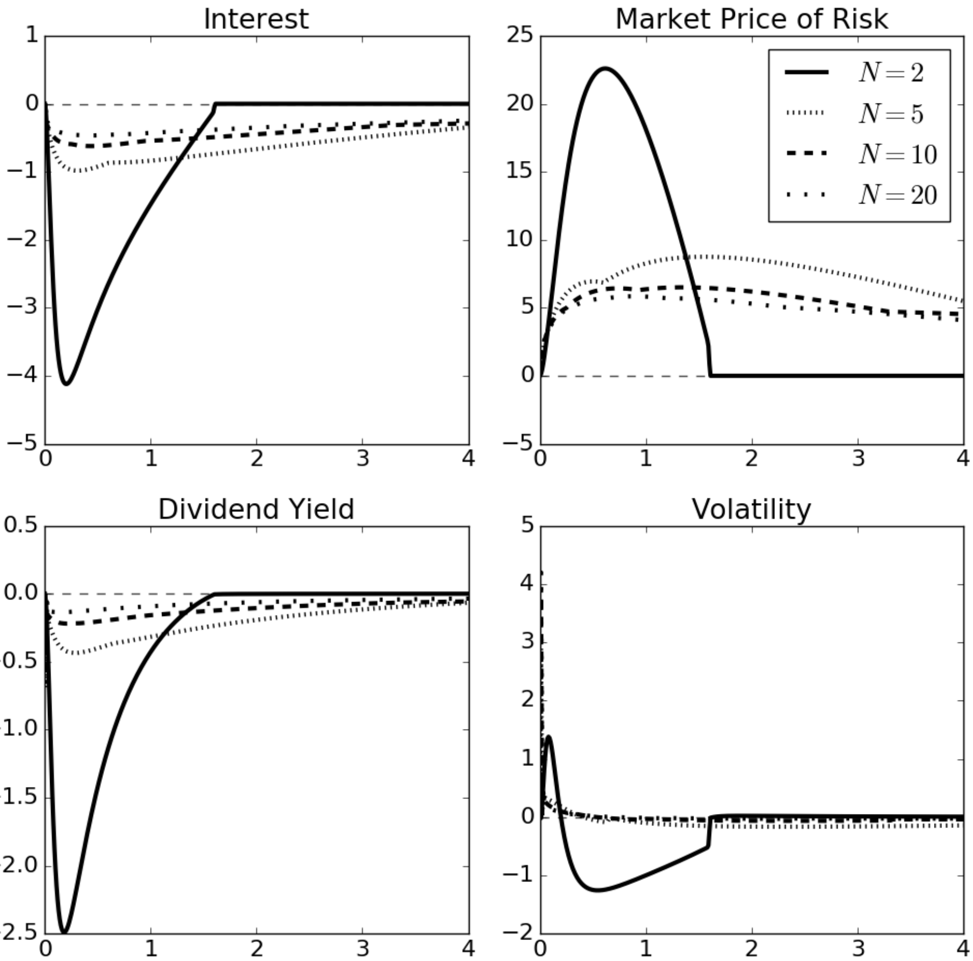


Fig. 7. Relative deviation of financial variables under margin constraint from complete market for 2, 5, 10, and 20 agents evenly distributed over $[1.1, 3.0]$.

presents the deviations from complete markets. The constrained region exhibits procyclical cycles over more states when there are more types. The level of leverage becomes more stable with more types, the peak of leverage is reduced, and the slope of leverage with respect to the state is smaller in absolute value. In this way, greater heterogeneity reduces the severity of the leverage cycle. As dividends fall, agents leverage up until they hit their constraint, at which point they begin to de-leverage. If a large mass hits this constraint at the same time, the resulting cycle will be greater, while if smaller groups are constrained at heterogeneous points in the state space the peak is reduced.

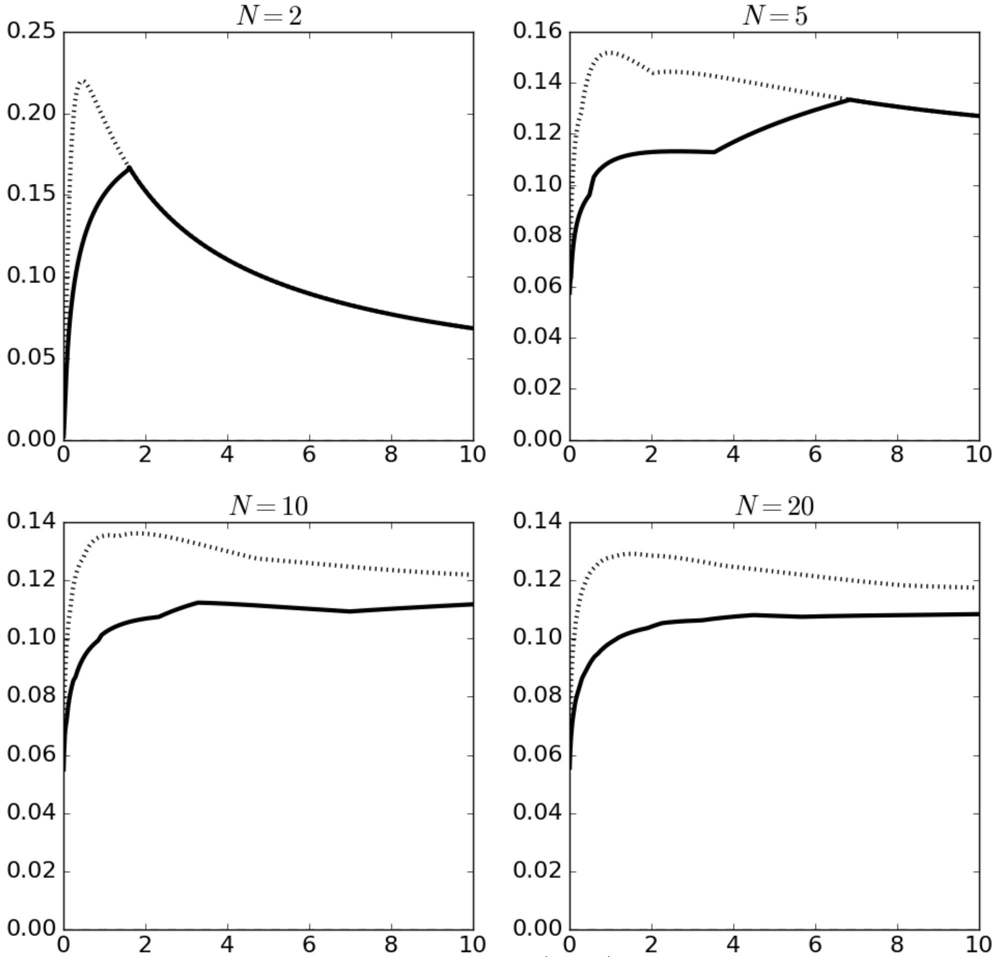


Fig. 8. Leverage under complete market (dots) and under margin constraint (solid) for 2, 5, 10, and 20 agents evenly distributed over $[1.1, 3.0]$.

The degree of heterogeneity has a substantial effect on model outcomes when agents face margin constraints. The combination of heterogeneity and margin constraints reconciles several facts about the financial market. With only two types there is a drop in volatility at the constraint threshold, while with many types this is reduced. In addition, this jump in volatility is associated with a change in the cyclicality of the leverage cycle. The market price of risk is greater when agents are constrained and this fact remains even when the number of types grows. There is a limit to arbitrage as risk neutral agents cannot borrow to profit from higher returns, producing a bubble whose size falls when there are many types, despite returns remaining high.

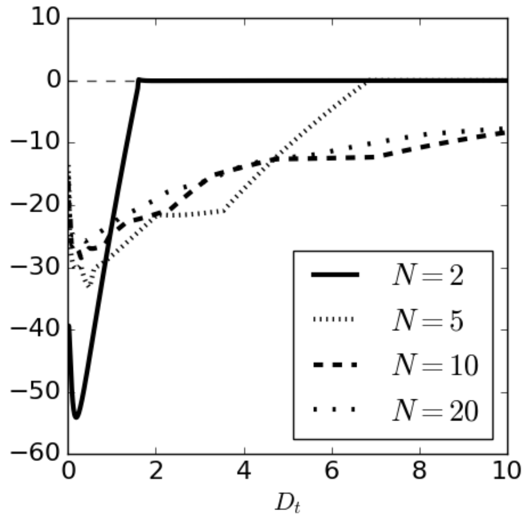


Fig. 9. Relative deviation of total leverage under margin constraint from complete market for 2, 5, 10, and 20 agents evenly distributed over $[1.1, 3.0]$.

Finally, leverage cycles become pro-cyclical and less severe, pointing towards possible empirical and policy implications.

5. Conclusion

In this paper I've documented a new stylized fact about the leverage cycle and proposed a model of preference heterogeneity and margin constraint to explain this fact. The equilibrium and solution method are novel, to my knowledge. The methodological contribution goes beyond the present setting to any model with homogeneous (in the functional sense) utility functions and incomplete markets. In particular the method could be applied to macroeconomic models such as Krusell and Smith (1998) when preferences are of the right type. Future work on this topic should build incrementally, introducing stochastic endowment and more general preferences. The economic contribution is to show how the degree of heterogeneity actually buffers some aspects of financial crises driven by margin constraints, but not others. In particular leverage cycles are less severe and credit contractions reduced when agents are more diverse, as well the interest rate, volatility, and asset prices deviate from complete markets to a lesser degree. However, the market price of risk remains far from its complete markets level as expected returns on risky assets must be higher in order to compensate risk averse agents.

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Appendix A. Proofs

Proof of Proposition 1. Take the market clearing condition for consumption:

$$D_t = \frac{1}{N}\sum_j c_{jt} = \frac{1}{N}\sum_j \left(\frac{c_{jt}^{-\gamma_j}}{c_{it}^{-\gamma_i}}\right)^{\frac{-1}{\gamma_j}} \left(\frac{c_{it}}{D_t}\right)^{\frac{\gamma_i}{\gamma_j}} D_t^{\frac{\gamma_i}{\gamma_j}} \tag{20}$$

Substitute Eq. (5) and re-arrange to find Eq. (6). Define the wealth/consumption ratio of an individual as $V_i(Y) = X_{it}/c_{it}$ for some vector of state variables Y. Take the market clearing condition in wealth

$$S_t = \frac{1}{N}\sum_j X_{jt} = \frac{1}{N}\sum_j V_j(Y)c_{jt} = \frac{1}{N}\sum_j V_j(Y)\left(\frac{c_{jt}^{-\gamma_j}}{c_{it}^{-\gamma_i}}\right)^{\frac{-1}{\gamma_j}} \left(\frac{c_{it}}{D_t}\right)^{\frac{\gamma_i}{\gamma_j}} D_t^{\frac{\gamma_i}{\gamma_j}}$$

Again substitute Eq. (5) into the last equality and re-arrange to find

$$0 = \frac{1}{N}\sum_j V_j(Y)\left[\left(\frac{\Lambda_j H_{jt}}{\Lambda_i H_{it}}\right)^{-\frac{1}{\gamma_j}} \omega_{it}^{\frac{\gamma_i}{\gamma_j}} D_t^{\frac{\gamma_i}{\gamma_j}-1} - \omega_{jt}\right] \tag{21}$$

Notice that Eq. (6) and Eq. (21), conditional on knowledge of the wealth consumption ratios form a system of $2N$ equations in $2N$ unknowns. We choose $Y = D$ to be the minimal state vector, implying that this system of equations implies the ratios of marginal utilities and consumption weights for every value of D. This is not to say that this is the only equilibrium, but the remainder of the paper calculates this equilibrium and proves that it is an optimum. □

Proof of Proposition 2. Transform the variables in Eqs. (6) and (7) by a normalization, such that

$$\omega_{it} = \hat{\omega}_{it}\sum_i \omega_{it} \tag{22}$$

$$H_{it} = \hat{H}_{it}\sum_i H_{it} \tag{23}$$

These new variables combined in a vector live on the $2N$ standard simplex:

$Z = (\hat{H}_{it}, \hat{\omega}_{it}) \in \Delta_{2N}$, which is a closed, convex set. Now substitute these variables into the equations in Eqs. (6) and (7) and define the resulting system as a vector value function $F(Z)$ such that $F(Z) = 0$. Finally, add Z to both sides and define a new function $G(\cdot)$ such that $G(Z) = F(Z) + Z = Z$. This implies that $G : \Delta_{2N} \to \Delta_{2N}$ satisfies the conditions of Brouwer's fixed point theorem and thus there exists at least one fixed point. □

Proof of Proposition 3. Take the market clearing condition in consumption and divide through by agent i's consumption

$$\frac{1}{N}\sum_j c_{jt} = D_t \Leftrightarrow c_{it} = \frac{c_{it}}{\frac{1}{N}\sum_j c_{jt}} D_t = \left(\frac{N\left(e^{\rho t}\Lambda_i H_{it}\right)^{\frac{-1}{\gamma_i}}}{\sum_j \left(e^{\rho t}\Lambda_j H_{jt}\right)^{\frac{-1}{\gamma_j}}} \right) D_t = \omega_{it} D_{it}$$

where ω_{it} represents an individual's consumption weight and is given by

$$\omega_{it} = \frac{N\left(e^{\rho t}\Lambda_i H_{it}\right)^{\frac{-1}{\gamma_i}}}{\sum_{j=1}^{N} \left(e^{\rho t}\Lambda_j H_{jt}\right)^{\frac{-1}{\gamma_j}}}$$

Assume individual consumption follows a GBM

$$\frac{dc_{it}}{c_{it}} = \mu_{cit} dt + \sigma_{cit} dW(t) \tag{24}$$

Apply Itô's lemma to Eq. (5) and solve for μ_{cit} and σ_{cit}

$$\mu_{cit} = \frac{r_t - \rho + \delta_i(\nu_{it})}{\gamma_i} + \frac{1+\gamma_i}{\gamma_i^2}\frac{1}{2}\left(\theta_t + \frac{\nu_{it}}{\sigma_t}\right)^2 \quad , \quad \sigma_{c^i}(t) = \frac{1}{\gamma_i}\left(\theta + \frac{\nu_{it}}{\sigma_t}\right)$$

Apply Itô's lemma to the market clearing condition for consumption and match coefficients to find

$$\mu_D = \frac{1}{N}\sum_{i=1}^{N} \omega_{it}\mu_{cit} \quad , \quad \sigma_D = \frac{1}{N}\sum_{i=1}^{N} \omega_{it}\sigma_{cit}$$

Now substitute the values for consumption drift and diffusion and solve for the interest rate and the market price of risk:

$$\theta_t = \frac{N}{\sum_i \frac{\omega_{it}}{\gamma_i}} \left(\sigma_D - \frac{1}{\sigma_t N} \sum_i \frac{\omega_{it}\nu_{it}}{\gamma_i} \right)$$

$$r_t = \frac{N}{\sum_i \frac{\omega_{it}}{\gamma_i}} \left(\mu_D + \frac{\rho}{N} \sum_i \frac{\omega_{it}}{\gamma_i} - \frac{1}{N} \sum_i \frac{\omega_{it}}{\gamma_i} \delta_i(\nu_{it}) - \frac{1}{2N} \sum_i \frac{1+\gamma_i}{\gamma_i} \left(\theta_t + \frac{\nu_{it}}{\sigma_t} \right)^2 \omega_{it} \right)$$

Next, apply Itô's lemma to $\omega_{it} = \frac{c_{it}}{D_t}$ and match coefficients to find the dynamics of consumption weights in Eq. (12) and Eq. (13).

□

Proof of Proposition 4. Assume there exists a Markovian equilibrium in D_t. Then an individual's Hamilton-Jacobi-Bellman (HJB) equation writes

$$0 = \max_{c_{it}, \pi_{it}} \left\{ e^{-\rho t} \frac{c_{it}^{1-\gamma_i}}{1-\gamma_i} + \frac{\partial J_{it}}{\partial t} + \left[X_{it} \left(r_t + \delta_i(\nu_{it}) + \pi_{it}\sigma_t \left(\theta_t + \frac{\nu_{it}}{\sigma_t} \right) \right) - c_{it} \right] \frac{\partial J_{it}}{\partial X_{it}} \right.$$
$$\left. + \mu_D D_t \frac{\partial J_{it}}{\partial D_t} + \sigma_D \sigma_t \pi_{it} D_t X_{it} \frac{\partial^2 J_{it}}{\partial X_{it} \partial D_t} + \frac{1}{2} \left[X_{it}^2 \pi_{.it}^2 \sigma_t^2 \frac{\partial^2 J_{it}}{\partial X_{it}^2} + \sigma_D^2 D_t^2 \frac{\partial^2 J_{it}}{\partial D_t^2} \right] \right\} \tag{25}$$

subject to the transversality condition $\mathbb{E}_t J_{it} \to 0$ for all i s.t. $\gamma_i > \underline{\gamma}$, as the agent with the lowest risk aversion will dominate in the long run (Cvitanić et al. (2011)). First order conditions imply

$$c_{it} = \left(e^{\rho t} \frac{\partial J_{it}}{\partial X_{it}} \right)^{\frac{-1}{\gamma_i}} \tag{26}$$

$$\pi_{it} = -\left(X_{it}\sigma_t \frac{\partial^2 J_{it}}{\partial X_{it}^2} \right)^{-1} \left[\left(\theta_t + \frac{\nu_{it}}{\sigma_t} \right) \frac{\partial J_{it}}{\partial X_{it}} + \sigma_D D_t \frac{\partial^2 J_{it}}{\partial X_{it} \partial D_t} \right] \tag{27}$$

Assume that the value function is separable as

$$J_{it}(X_{it}, D_t) = e^{-\rho t} \frac{X_{it}^{1-\gamma_i} V_i(D)^{\gamma_i}}{1-\gamma_i} \tag{28}$$

Substituting Eq. (28) into Eqs. (26) and (27) gives

$$c_{it} = \frac{X_{it}}{V_i(D)} \tag{29}$$

$$\pi_{it} = \frac{1}{\gamma_i \sigma_t}\left(\gamma_i \sigma_D D_t \frac{V_i'(D)}{V_i(D)} + \theta_t + \frac{\nu_{it}}{\sigma_t}\right) \tag{30}$$

which shows that $V_i(D)$ is the wealth-consumption ratio as a function of the dividend. Next, substitute Eqs. (28) to (30) into Eq. (25) and simplify to find

$$\begin{aligned} 0 =& 1 + \frac{\sigma_D^2 D^2}{2} V_i''(D) + \left[\frac{1-\gamma_i}{\gamma_i}\left(\theta_t + \frac{\nu_{it}}{\sigma_t}\right)\sigma_D + \mu_D\right] D V_i'(D) \\ &+ \left[(1-\gamma_i)(r_t - \delta_i(\nu_{it})) - \rho + \frac{1-\gamma_i}{2\gamma_i}\left(\theta_t + \frac{\nu_{it}}{\sigma_t}\right)^2\right]\frac{V_i(D)}{\gamma_i} \end{aligned} \tag{31}$$

which gives an ode for the wealth-consumption ratio over the state space. The boundary conditions are given by recognizing that the limit in $D \to \infty$ is an economy in autarky dominated by the most risk neutral agent (Cvitanić et al. (2011); Chabakauri (2015)). Similarly, as $D \to 0$ the most risk averse agent dominates.

Define the price-dividend ratio as a function of the single state variable: $\mathcal{S}(D_t) = \frac{S_t}{D_t}$. Apply Itô's lemma to $D_t\mathcal{S} = S_t$ and match coefficients to find

$$\mu_t = D_t^2 + \frac{(\sigma_D D_t)^2}{2}\frac{\mathcal{S}''(D_t)}{\mathcal{S}(D_t)} D_t + D_t \mu_D + \frac{\mathcal{S}'(D_t)}{\mathcal{S}(D_t)}(\sigma_D D_t)^2$$

$$\sigma_t = \sigma_D\left(1 + D_t \frac{\mathcal{S}'(D_t)}{\mathcal{S}(D_t)}\right)$$

Taking the market clearing condition for wealth, rewrite $\mathcal{S}(D_t)$ as a function of D_t:

$$S_t = \frac{1}{N}\sum_i X_{it} \Leftrightarrow \frac{S_t}{D_t} = \mathcal{S}(D_t) = \frac{1}{N}\sum_i \frac{X_{it}}{D_t} = \frac{1}{N}\sum_i \frac{X_{it}}{c_{it}}\frac{c_{it}}{D_t} = \frac{1}{N}\sum_i V(D_t)\omega_{it}$$

which gives $\mathcal{S}(D_t)$ given that $\omega_{it} = f_i(D_t)$ □

Proof of Proposition 5. For a homogeneous margin constraint, $\nu_{it} \leq 0$ and $m \geq 0$, thus $\nu_{it} m \leq 0$ (Cvitanić and Karatzas (1992); Chabakauri (2015)). Additionally, $\pi_{it} \leq m$. Substituting the solution for π_{it} from Eq. (30) into the latter inequality and recognizing that, by the Kuhn-Tucker conditions at least one of the inequalities holds with equality gives the result. □

Proof of Proposition 6. This proof proceeds identically to Chabakauri (2015). Let $V_i(D) \in C^1[0,\infty) \cup C^2[0,\infty)$, $0 < V_i(D) \leq C_1$, $|\pi_{it}\sigma_t| < C1$, and $|\theta_t + \nu_{it}/\sigma_t| < C_1$, where C_1 is a constant. Additionally, assume

$$\mathbb{E} \int_0^\infty e^{-\rho t} \frac{c_{it}^{1-\gamma_i}}{1-\gamma_i} dt < \infty \tag{32}$$

$$\mathbb{E} \int_0^T J_i(X_{it}, D_t, t)^2 dt < \infty \qquad \forall\ T > 0 \tag{33}$$

$$\limsup_{T\to\infty} \mathbb{E} J_i(X_{it}, D_t, t) \geq 0 \tag{34}$$

Define $U_t = \int_0^t e^{-\rho t} c_{i\tau}^{1-\gamma_i}/(1-\gamma_i) d\tau + J_i(X_{it}, D_t, t)$, which satisfies $dU_t = \mu_{Ut} dt + \sigma_{Ut} dW_t$ such that

$$\begin{aligned}
\mu_{Ut} = \Bigg(& e^{-\rho t} \frac{c_{it}^{1-\gamma_i} - 1}{1-\gamma_i} + \frac{\partial J_{it}}{\partial t} + \left[X_{it} \left(r_t + \delta(\nu_{it}) + \pi_{it}\sigma_t \left(\theta_t + \frac{\nu_{it}}{\sigma_t} \right) \right) - c_{it} \right] \frac{\partial J_{it}}{\partial X_{it}} \\
& + \mu_D D_t \frac{\partial J_{it}}{\partial D_t} + \sigma_D \sigma_t \pi_{it} D_t X_{it} \frac{\partial^2 J_{it}}{\partial X_{it} \partial D_t} + \frac{1}{2} \left[X_{it}^2 \pi_{.it}^2 \sigma_t^2 \frac{\partial^2 J_{it}}{\partial X_{it}^2} + \sigma_D^2 D_t^2 \frac{\partial^2 J_{it}}{\partial D_t^2} \right] \Bigg) \\
& - (\nu_{it}\pi_{it} + \delta(\nu_{it})) X_{it} \frac{\partial J_{it}}{\partial X_{it}}
\end{aligned} \tag{35}$$

$$\sigma_{Ut} = J_{it} \left((1-\gamma_i)\pi_{it}\sigma_t + \gamma_i D_t \sigma_D \frac{V_i'(D)}{V_i(D)} \right) = J_{it} \left(\pi_{it}\sigma_t - \theta_t - \frac{\nu_{it}}{\sigma_t} \right) \tag{36}$$

The first term in μ_{Ut} is simply the PDE inside the max operator in Eq. (25), and is thus weakly negative. The second term is as well, as $\nu_{it}\pi_{it} + \delta(\nu_{it}) \geq 0$ by definition and $\partial_{X_{it}} J_{it} \geq 0$. Thus $\mu_{Ut} \leq 0$. By the boundedness conditions, U_t is integrable and because its drift is negative it is a supermartingale, thus $U_t \geq \mathbb{E}_t U_T\ \forall\ t \leq T$, which is equivalent to

$$J_i(X_{it}, D_t, t) \geq \mathbb{E}_t \int_t^T e^{-\rho(\tau - t)} \frac{c_{i\tau}^{1-\gamma_t}}{1-\gamma_i} d\tau + \mathbb{E}_t J_i(X_{it}, D_t, T) \tag{37}$$

Since the first term is monotonic in T, by Eq. (34) and by the monotone convergence theorem we have

$$J_i(X_{it}, D_t, t) \geq \mathbb{E}_t \int_t^{\infty} e^{-\rho(\tau - t)} \frac{c_{i\tau}^{1-\gamma_t}}{1-\gamma_i} d\tau \tag{38}$$

Now to show the opposite, we first show that $\mathbb{E}_t J_i(X_{i\tau}, D_\tau, \tau) \to 0$ as $\tau \to \infty$. Applying Itô's lemma to $J_i(X_{it}, D_t, t)$ and following similar steps as before, we find $dJ_{it} = J_{it}[\mu_{Jt} dt + \sigma_{Jt} dW_t]$ where

$$\begin{aligned} \mu_{Jt} &= \frac{-1}{V_i(D)} \\ \sigma_{Jt} &= \pi_{it}\sigma_t - \theta_t - \frac{\nu_{it}}{\sigma_t} \end{aligned}$$

by the first order conditions Eqs. (29) and (30). By the boundedness assumptions σ_{Jt} satisfies Novikov's conditions and we have that $d\eta_t = \eta_t \sigma_{Jt} dW_t$ acts as a change of measure to remove the Brownian term in J_{it}. We have

$$\begin{aligned} |\mathbb{E}_t J_i(X_{i\tau}^*, D_\tau, \tau)| &\leq \mathbb{E}_t \left[|J_{i\tau}| exp \left\{ - \int_t^T \frac{1}{V_i(D)} du \right\} \frac{\eta_\tau}{\eta_t} \right] \\ &\leq |J_{it}| e^{-(T-t)/C_1} \mathbb{E}_t \frac{\eta_\tau}{\eta_t} = |J_{it}| e^{-(T-t)/C_1} \end{aligned}$$

Taking the limit in T gives the result.

Finally, define U_t^* as for U_t, except evaluated at the optimum consumption. Then

$$dU_t^* = J_{it} \left(\pi_{it}\sigma_t - \theta_t - \frac{\nu_{it}}{\sigma_t} \right) dW_t \tag{39}$$

Again applying Novikov's condition we get that U_t^* is an exponential martingale, which gives (after integrating Eq. (39))

$$J_i(X_{it}, D_t, t) = \mathbb{E}_t \int_t^T e^{-\rho(\tau - t)} \frac{(c_{it}^*)^{1-\gamma_t}}{1-\gamma_i} d\tau + \mathbb{E}_t J_i(X_{it}^*, D_t, T)$$

Finally, by the intermediate result the last term goes to zero, showing that we do indeed have an optimum. □

Appendix B. Numerical Method

The problem presented by the equilibrium under margin constraints has several difficult features which make it challenging from a numerical perspective. First, the system of ODE's in Eq. (14) represents a highly non-linear system, as the coefficients depend in a non-trivial way on the solution itself. Second, Eq. (6) represents a set of constraints on the solution to the ODE's. These two facts combined place the problem under the framework of "Non-Linear Differential Algebraic Systems". Finally, at the point where an individual's portfolio constraint binds there will exist a singularity. The consumption weights will be kinked (and thus not differentiable) creating a jump in the coefficients. This phase transition creates a free boundary problem, as it is impossible to determine analytically the point at which the constraints bind. All of these points combined make this problem particularly challenging (for more on the mathematical particularities of these topics see Wanner and Hairer (1991); Hirsch et al. (2012); Friedman (1982)).

Luckily, the numerical solution exhibits characteristics which make an ad-hoc solution algorithm possible. First, the equations seem to be hyperbolic (LeVeque (2002)), implying that an implicit-explicit backwards Euler approach requires only the terminal condition and avoids noise introduced by the Dirichlet boundary conditions in Eq. (15). In addition, although the constrained consumption weights differ from the unconstrained case, they do so only slightly, making an iterative scheme relatively stable. Lastly, although the coefficients exhibit jumps the solution to the ODE's are smooth and thus a classical finite difference method suffices and one can avoid the use of viscosity solutions (although upwinding seems to add stability).

To solve the system of ODE's I use a finite difference approach. Given that the system is non-linear and must satisfy a constraint, I apply a single Picard type step by solving the ODE's, using these to calculate consumption weights, and again solving the ODE's.

Discretize the system of ODE's using an implicit-explicit scheme:

$$\frac{V_{ik}^m - V_{ik}^{m-1}}{\Delta_t} + 1 + a^m(D_k)\frac{V_{ik-1}^m - 2V_{ik}^m + V_{ik+1}^m}{\Delta_D^2} + b^m(D_k)\frac{V_{ik-1}^m - V_{ik+1}^m}{2\Delta_D} + c(D_k)V_{ik}^m = 0$$

1. Fix an initial distribution for the ω_{it}'s, assuming the complete markets solution.
2. To simplify computations, assume initial wealth is such that $\Lambda_i = 1$ for all i (This could be relaxed, but would require Monte Carlo to approximate these Lagrange multipliers).
3. Fix a grid of M points over (D).
4. Fix an initial guess for $V_i(D) = \Delta_t$, the size of the time step, representing the long run value.
5. Iterate over the following until convergence:
   (a) Given V_{ik}^m, calculate implied values for r_{ik}^m, θ_{ik}^m, σ_{ik}^m, ν_{ik}^m, which in turn imply values for $a^m(D_k)$, $b^m(D_k)$, $c^m(D_k)$.
   (b) Solve the discretized version of the ODE for V_i^{m-1}.
6. Upon convergence, solve the system of equations in Eqs. (6) and (7) for ω_{it}.
7. Again solve backwards the ODE's until convergence. And stop.